Intersection Types and Topologies in Lambda Calculus

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Intersection Types and Related Systems
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Outline of the talk:

- Full intersectioin type system $\lambda\cap_\Omega$
- Topologies on the set Λ of untyped lambda terms
- Continuity Theorem
- Topological characterizations:
  - Unsolvable terms
  - Normal forms
  - $\beta\eta$-normal forms
- Specialization preorder
- Related work
- Comparing topologies
- Discussion

$\boxed{\Lambda}$ Set of untyped lambda terms

1. *Var* $\subseteq \Lambda$ (variables)

2. $M, N \in \Lambda \Rightarrow MN \in \Lambda$ (app)

3. $x \in$ *Var*, $M \in \Lambda \Rightarrow \lambda x.M \in \Lambda$ (abs)

$\boxed{\textit{type}}$ Set of intersection types

1. $\Omega \in$ *type*

2. *atom* $\subseteq$ *type*

3. $\varphi, \psi \in$ *type* $\Rightarrow \varphi \to \psi \in$ *type*

4. $\varphi, \psi \in$ *type* $\Rightarrow \varphi \cap \psi \in$ *type*

$\boxed{\leq}$ Preorder on *type*

1. $\sigma \leq \sigma$
2. $\sigma \leq \tau, \quad \tau \leq \rho \Rightarrow \sigma \leq \rho$
3. $\sigma \leq \sigma \cap \sigma$
4. $\sigma \cap \tau \leq \sigma, \quad \sigma \cap \tau \leq \tau$
5. $\sigma \leq \tau, \quad \sigma_1 \leq \tau_1 \Rightarrow \sigma \cap \sigma_1 \leq \tau \cap \tau_1$
6. $(\sigma \to \rho) \cap (\sigma \to \tau) \leq \sigma \to (\rho \cap \tau)$
7. $\sigma \leq \sigma_1, \quad \tau \leq \tau_1 \Rightarrow \sigma_1 \to \tau \leq \sigma \to \tau_1$
8. $\sigma \leq \Omega$
9. $\Omega \leq \Omega \to \Omega$

$\boxed{\lambda\cap_\Omega}$ Intersection type system

$$(axiom) \quad \frac{(x:\sigma) \in \Gamma}{\Gamma \vdash x:\sigma}$$

$$(\to E) \quad \frac{\Gamma \vdash M:\sigma \to \tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash MN:\tau}$$

$$(\to I) \quad \frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash (\lambda x.M):\sigma \to \tau}$$

$$(\cap I) \quad \frac{\Gamma \vdash M:\sigma \quad \Gamma \vdash M:\tau}{\Gamma \vdash M:\sigma \cap \tau}$$

$$(\Omega) \quad \frac{}{\Gamma \vdash M:\Omega}$$

$$(\leq) \quad \frac{\Gamma \vdash M:\sigma \quad \sigma \leq \tau}{\Gamma \vdash M:\tau}$$

Topologies on Λ

$$\mathcal{V}_{\Gamma,\sigma} = \{M \in \Lambda \mid \Gamma \vdash M : \sigma\}.$$

- $\mathcal{V}_{\Gamma,\Omega} = \Lambda$.
- $\mathcal{V}_{\Gamma,\sigma} \cap \mathcal{V}_{\Gamma,\tau} = \mathcal{V}_{\Gamma,\sigma\cap\tau}$.
- $\mathbf{B}_\Gamma = \{\mathcal{V}_{\Gamma,\sigma}\}_{\sigma \in type}$ basis of a topology on Λ:

1. $M \in \Lambda \;\; \exists \mathcal{V}_{\Gamma,\sigma} \in \mathbf{B}_\Gamma \;\; M \in \mathcal{V}_{\Gamma,\sigma}$
2. $\mathcal{V}_{\Gamma,\sigma}, \;\; \mathcal{V}_{\Gamma,\tau} \in \mathbf{B}_\Gamma \; \exists X \in \mathbf{B}_\Gamma$

   $X \subseteq \mathcal{V}_{\Gamma,\sigma} \cap \mathcal{V}_{\Gamma,\tau} \; (X = \mathcal{V}_{\Gamma,\sigma\cap\tau})$

Continuity Theorem

$$App : \Lambda \times \Lambda \to \Lambda$$

1. $App(F, M) = FM$ is continuous w.r.t. each Γ-fit topology

$$Abs : \mathit{Var} \times \Lambda \to \Lambda$$

2. $Abs(x, M) = \lambda x.M$ is continuous w.r.t. each Γ-fit topology

Proof

$$(\forall \mathcal{V}_{\Gamma,\varepsilon} \ni FM)(\exists \mathcal{V}_{\Gamma,\delta_1} \ni F)(\exists \mathcal{V}_{\Gamma,\delta_2} \ni M)$$

$$(G \in \mathcal{V}_{\Gamma,\delta_1} \wedge N \in \mathcal{V}_{\Gamma,\delta_2} \Rightarrow GN \in \mathcal{V}_{\Gamma,\varepsilon})$$

$$\Gamma \vdash FM : \varepsilon \Rightarrow \Gamma \vdash F : \delta \to \varepsilon \;\; \Gamma \vdash M : \delta$$

Taking $\delta_1 \equiv \delta \to \varepsilon$ and $\delta_2 \equiv \delta$ we prove the statement

Topological properties of reductions

(i) If $M \twoheadrightarrow_\beta P$ then

$$M \in \mathcal{V}_{\Gamma,\sigma} \Leftrightarrow P \in \mathcal{V}_{\Gamma,\sigma}$$

(ii) If $M \twoheadrightarrow_\eta P$ then

$$(M \in \mathcal{V}_{\Gamma,\sigma} \Rightarrow P \in \mathcal{V}_{\Gamma,\sigma})$$

(iii) If $N \in \Lambda$ is a $\beta\eta$–nf then

$$M \twoheadrightarrow_\beta N \ \text{ iff } \ (M \in \mathcal{V}_{\Gamma,\sigma} \Leftrightarrow N \in \mathcal{V}_{\Gamma,\sigma})$$

Topological characteristics of terms

(i) Unsolvable terms are bottoms (generic points) w.r.t. each Γ-fit topology.

(ii) If N is a nf, then for some Γ-fit topology $\mathcal{V}_{\Gamma,\rho} = \{M \mid N \twoheadrightarrow_\eta M\}$ (Principal typing)

(iii) $\beta\eta$-nf is an isolated point w.r.t. some Γ-fit topologies $\mathcal{V}_{\Gamma,\rho} = \{N\}$ (singleton)

Genericity lemma

M unsolvable term and let N nf

$$FM =_\beta N \Rightarrow \forall L \in \Lambda \ \ (FL =_{\beta\eta} N)$$

A natural question:

How to restrict all Γ-fit topologies to the set Λ^0 of closed lambda terms?

Unique way

All Γ-fit topologies collapse into one topology with basis

$$\mathbf{B}_{\emptyset} = \{\mathcal{V}_\sigma \mid \sigma \in type\}$$

$$\mathcal{V}_\sigma = \{M \in \Lambda^0 \mid \ \vdash M : \sigma\}$$

$$\mathcal{V}_\sigma \subseteq \mathcal{V}_{\Gamma,\sigma}$$

Specialization preorder on Λ

(i) $M \sqsubseteq N$ *iff* every open set containing M also contains N

(ii) $M \sim N$ *iff* $M \sqsubseteq N$ and $N \sqsubseteq M$

Reductions and preorder

1. If $M \twoheadrightarrow_\beta N$, then $M \sim N$
2. If $M \twoheadrightarrow_\eta N$, then $M \sqsubseteq N$
3. Characterization (partial) of $\sim$
   - N $\beta\eta$-nf
     $$M \sim N \;\; \textit{iff} \;\; M \twoheadrightarrow_\beta N$$
   - M, N normalizing
     $$M \sim N \;\; \textit{iff} \;\; M =_\beta N$$
   - M unsolvable
     $$M \sim N \; \textit{iff} \; N \text{ unsovable}$$

4. Characterization (partial) of $\sqsubseteq$

– M normalizing (to M_{nf})

$$M \sqsubseteq N \;\; \textit{iff} \;\; M_{nf} \twoheadrightarrow_{\eta} P \quad N \twoheadrightarrow_{\beta} P$$

– M unsolvable

$$\forall N \in \Lambda \quad M \sqsubseteq N$$

Filter topology

$\mathcal{F}$ is a complete lattice under set theoretical inclusion
Scott topology in a standard way.

$\mathcal{O} \subseteq \mathcal{F}$ is open:

(i) $d \in \mathcal{O}$ and $d \subseteq e \Rightarrow e \in \mathcal{O}$;

(ii) $\cup_{i \in I} d_i \in \mathcal{O} \Rightarrow \exists i_0 \in I \ d_{i_0} \in \mathcal{O}$.

A basis for the Scott topology on $\mathcal{F}$ is the set of upper closures of principal filters $\{\uparrow \widehat{\sigma} \mid \sigma \in type\}$

A *valuation of lambda terms in filters* $\| \ \|_\Gamma : \Lambda \to \mathcal{F}$

$$\|M\|_\Gamma = \{\sigma \mid \Gamma \vdash M : \sigma\} \in \mathcal{F}$$

depends on Γ not unique

Γ-filter topology on Λ is the smallest that makes $\| \quad \|_\Gamma$ continuous w.r.t. the Scott topology on $\mathcal{F}$

Open sets in Λ are obtained from open sets in $\mathcal{F}$ by the inverse map.

For every $\Gamma \in \mathbf{E}$ the family

$$\{\| \uparrow \widehat{\sigma}\|_{\Gamma}^{-1}\}_{\sigma \in type}$$

is a basis for the Γ-filter topology on Λ.

Comparing topologies

Γ-fit and Γ-filter topologies on Λ coincide.

$$\| \uparrow \widehat{\sigma}\|_{\Gamma}^{-1} = \mathcal{V}_{\Gamma,\sigma}$$

Discussion

Tree topology $BT : \Lambda \to \mathcal{B}$ continuous w.r.t. Scott topology on $\mathcal{B}$

- β-equal terms not separable

  same Böhm trees
  typeable by same types

- η-equal terms separable

• Indirect way :

  filter topology
  tree topology

• Direct way:

  fit topology