Full intersection types and topologies in lambda calculus [1]

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Abstract

Topologies are introduced on the set of lambda terms by their typeability in the full intersection type assignment system. These topologies give rise to simple proofs of some fundamental results of the lambda calculus such as the Continuity Theorem and the Genericity Lemma. We show that application is continuous, unsolvable terms are bottoms and normal forms are isolated points with respect to these topologies. The restriction of all these topologies to the set of closed lambda terms appears to be unique. We compare the introduced topology with the filter topology on the set of (closed) lambda terms and show that they coincide.

1. Introduction

Simply typed lambda calculus, $\lambda\!\to$, types terms from a subclass of the class of strongly normalizing lambda terms. It is well-known that $\lambda\!\to$ can be extended in various ways. One of these extensions is with intersection types (Coppo et al. [1980], Krivine [1990]). Types inhabited in the simply typed lambda calculus $\lambda\!\to$ correspond, by the Curry–Howard isomorphism, to the formulae provable in the implicational fragment of intuitionistic logic. Intersection types are often compared to conjunctive formulas albeit their logical interpretation given in Dezani–Ciancaglini et al. [1997] is different from intuitionistic conjunction.

The pure intersection type assignment system types all and only the strongly normalizing lambda terms (Krivine [1990], G. [1996]). Intersection types can be extended by a type ω, which can be assigned to every lambda term. This is how the full intersection type assignment system types all lambda terms. The completeness of the intersection type assignment system is shown in Barendregt et al. [1983] by using filter models. Filter models are developed on the notion of filters, special subsets of the set of all intersection types.

The known topologies in lambda calculus are the tree topology and the filter topology. The *tree topology* on the set of lambda terms is the smallest that makes the mapping from lambda terms to Böhm-like trees continuous w.r.t. the Scott topology on Böhm-like trees (see Barendregt [1984]). Thus, the tree topology consists of the inverse images under this mapping of the Scott-open sets of Böhm-like trees. The *filter topology* on the set of lambda terms is introduced in a similar indirect way. Namely, it is the smallest that makes the mapping from closed lambda terms to filters continuous w.r.t. the Scott topology on filters. Thus, the filter topology consists of the inverse images under this mapping of the Scott-open sets of filters.

The topologies here considered on the set Λ of all lambda terms are introduced directly, using basic open sets related to typing statements in the full intersection type assignment system. The author feels that the description of these topologies is simpler than those of the known ones (tree topology and filter topology) and leads to the same fundamental theorems of the untyped lambda calculus such as the Continuity Theorem and the Genericity Lemma (see Barendregt [1984] Chapter 14). As soon as we have topologized Λ, we get the specialization preorder and the corresponding equivalence relation on Λ (see Vickers [1989]). We give a characterization of these relations in terms of β- and η-reductions.

Section 2 is an overview of some basic notions and known properties of the untyped lambda calculus and intersection type assignment systems. In Section 3 topologies, the Γ-*fit topology* are defined on the set Λ of all lambda terms and some of their properties are investigated. The restriction of these topologies to the set Λ^0 of closed lambda terms, the *fit topology*, is considered further. In Section 4 we compare the fit topology and the filter topology and show that:

- they coincide on the set Λ^0 of closed lambda terms,
- for every Γ-fit topology on the set Λ of all lambda terms there is a coincident filter topology on Λ, and vice versa.

2. Intersection type assignment systems

Definition 2.1 *The set Λ of lambda terms is defined by the following abstract syntax.*

Λ = `var` $\mid$ $\Lambda\Lambda$ $\mid$ λ`var`.Λ

`var` = x $\mid$ `var'`.

We write $x,\ y,\ z,\ \ldots$ for arbitrary term variables and $M,\ N,\ P,\ Q, \ldots$ for arbitrary terms.

The set $Fv(M)$ of *free variables* of a term M is defined in the usual way. The set Λ^0 of *closed lambda terms* is the set of lambda terms with no free variables

$$\Lambda^0 = \{P \in \Lambda \mid Fv(P) = \emptyset\}.$$

$\equiv$ denotes the syntactical equality of lambda terms up to renaming bound variables. The main axiom of *β-reduction* is $(\lambda x.M)N \to_\beta M[N/x]$, where $M[N/x]$ is obtained from M by substituting N for each free occurrence of x in M, taking into account that free variables of N remain free in the term obtained. $\to_\beta$ is closed under congruence. A term of the form $(\lambda x.M)N$ is called a *redex*. The transitive reflexive closure of $\to_\beta$ is denoted by $\twoheadrightarrow_\beta$. *β-equality*, $=_\beta$, is the least equivalence relation containing $\twoheadrightarrow_\beta$.

A *normal form* is a term which does not contain any redex. If $P \twoheadrightarrow_\beta N$ and N is a normal form, then we say that P is *normalizing* (*has a normal form*). Normal form of a term is unique, which is a consequence of the Church–Rosser Theorem. A term is *strongly normalizing* if all its reduction paths end in a normal form. A *head normal form* is a term of the form $\lambda x_1 \ldots x_n.yM_1 \ldots M_k$. A term M is *solvable* (*has a head normal form*) if $M \twoheadrightarrow_\beta P$ and P is a head normal form. A term is *unsolvable* if it is not solvable.

The main axiom of *η-reduction* is $\lambda x.Mx \to_\eta M$, where $x \notin Fv(M)$. $\to_\eta$ is closed under congruence. A term of the form $\lambda x.Mx$, where $x \notin Fv(x)$, is called an *η-redex*. The transitive reflexive closure of $\to_\eta$ is $\twoheadrightarrow_\eta$. *η-equality*, $=_\eta$, is the least equivalence relation containing $\twoheadrightarrow_\eta$.

It is easy to prove that normal forms are preserved under η-reduction.

Proposition 2.2 *If N is a normal form and $N \twoheadrightarrow_\eta N'$, then N' is a normal form.*

A *$\beta\eta$-normal form* is a term which does not contain any redex nor any η-redex.

Definition 2.3 *The set* `type` *of types is defined by the following abstract syntax.*

`type` = ω $\mid$ `atom` $\mid$ `type` $\to$ `type` $\mid$ `type` $\cap$ `type`

`atom` = α $\mid$ `atom'`.

We write $\alpha,\ \beta,\ \gamma,\ \ldots \alpha_1, \ldots$ for arbitrary atoms and $\varphi,\ \rho,\ \sigma,\ \tau,\ \ldots\ \varphi_1,\ \ldots$ for arbitrary types.

Definition 2.4 *(i) A pre–order $\leq$ is the least relation on* `type` *satisfying the following:*

1. $\sigma \leq \sigma$	*5.* $(\sigma \to \rho) \cap (\sigma \to \tau) \leq \sigma \to (\rho \cap \tau)$
2. $\sigma \leq \tau, \quad \tau \leq \rho \Rightarrow \sigma \leq \rho$	*6.* $\sigma \cap \tau \leq \sigma, \quad \sigma \cap \tau \leq \tau$
3. $\sigma \leq \omega$	*7.* $\sigma \leq \tau, \quad \sigma \leq \rho \Rightarrow \sigma \leq \tau \cap \rho$
4. $\omega \leq \omega \to \omega$	*8.* $\sigma \leq \sigma_1, \quad \tau \leq \tau_1 \Rightarrow \sigma_1 \to \tau \leq \sigma \to \tau_1$.

(ii) $\sigma \sim \tau$ *if and only if* $\sigma \leq \tau$ *and* $\tau \leq \sigma$.

Definition 2.5 *(i) A* type assignment *is an expression of the form* $P : \sigma$, *where* P *is a lambda term and* σ *is a type.*

(ii) A declaration *is a type assignment of the form* $x : \rho$.

(iii) An environment Γ *is a set of declarations, such that all term variables are different.*

The set of all environments will be denoted by $\mathbf{E}$.

Definition 2.6 *A type assignment* $P : \sigma$ *is* derivable *from the environment* Γ *in the full intersection type assignment system* $\lambda\cap$, *notation*

$$\Gamma \vdash P : \sigma,$$

if $\Gamma \vdash P : \sigma$ *can be generated by the following axiom-scheme and rules:*

$$(axiom) \quad \frac{(x : \sigma) \in \Gamma}{\Gamma \vdash x : \sigma};$$

$$(\to E) \quad \frac{\Gamma \vdash M : \sigma \to \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}; \quad (\to I) \quad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash (\lambda x.M) : \sigma \to \tau};$$

$$(\cap E) \quad \frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \sigma}, \qquad \frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \tau}; \quad (\cap I) \quad \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \cap \tau};$$

$$(\omega) \quad \frac{}{\Gamma \vdash M : \omega};$$

$$(\leq) \quad \frac{\Gamma \vdash M : \sigma \quad \sigma \leq \tau}{\Gamma \vdash M : \tau}.$$

Some of the type assignment systems that can be obtained by combining the rules above and can be regarded as restrictions of $\lambda\cap$ are:

- $\lambda\!\to$, *simply typed lambda calculus*: $(axiom)$, $(\to E)$ and $(\to I)$;
- $\mathcal{D}$, *pure intersection type assignment system*: $(axiom)$, $(\to E)$, $(\to I)$, $(\cap E)$ and $(\cap I)$.

The simply typed lambda calculus, $\lambda\!\to$, assigns types to terms from a subclass of the class of strongly normalizing lambda terms. There are even normal forms which are not typeable in $\lambda\!\to$, e.g. $\lambda x.xx$. This term is typeable in $\mathcal{D}$, and therefore in all other intersection type systems

$$\vdash \; \lambda x.xx : (\sigma \cap (\sigma \to \tau)) \to \tau.$$

The pure intersection type assignment system $\mathcal{D}$ types exactly all strongly normalizing lambda terms (see Krivine [1990] and G. [1996]). The system $\lambda\cap$ types all lambda terms.

Some examples of typeability in $\lambda\cap$:

$\vdash \Omega : \omega$, where $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ (unsolvable term);

$\vdash \mathsf{Y} : (\omega \to \alpha) \to \alpha$, where $\mathsf{Y} \equiv \lambda f.[(\lambda x.f(xx))(\lambda x.f(xx))]$ (solvable term);

$\vdash \mathsf{K}\mathsf{I}\Omega : \alpha \to \alpha$, where $\mathsf{K}\mathsf{I}\Omega \equiv (\lambda xy.x)(\lambda x.x)\Omega$ (normalizing term).

There is a neat characterization of solvable, and hence of unsolvable terms, as duals, by their typeability in the full intersection type assignment system $\lambda\cap$.

Theorem 2.7 *(Barendregt et al. [1983], Theorem 4.13.)*

(i) M is solvable $\Leftrightarrow \exists \Gamma \in \mathbf{E}\ \ \exists \sigma \in \mathtt{type}\ (\Gamma \vdash M : \sigma \ \ \wedge \ \ \sigma \not\sim \omega)$.

(ii) M is unsolvable $\Leftrightarrow \forall \Gamma \in \mathbf{E}\ \ \forall \sigma \in \mathtt{type}\ (\Gamma \vdash M : \sigma \Rightarrow \sigma \sim \omega)$.

Subject reduction holds for both β- and η-reduction in $\lambda\!\to$ and $\lambda\cap$. Moreover, $\lambda\cap$ is closed for β-expansion, but not for η-expansion. Therefore the following properties hold for $\lambda\cap$.

Theorem 2.8 *(Barendregt et al. [1983], Corollary 3.8)*
Let $M, N \in \Lambda$.

(i) If $M \twoheadrightarrow_\beta N$, then $(\Gamma \vdash M : \sigma \Leftrightarrow \Gamma \vdash N : \sigma)$.

(ii) If $M \twoheadrightarrow_\eta N$, then $(\Gamma \vdash M : \sigma \Rightarrow \Gamma \vdash N : \sigma)$.

The following example illustrates that $\lambda\cap$ is not closed for η-expansion: $\mathsf{1} \equiv \lambda xy.xy \to_\eta \lambda x.x \equiv \mathsf{I}$, but there are types of I, which are not types of $\mathsf{1}$, such as $\alpha \to \alpha$, where α is an atomic type, since

$$\vdash \lambda x.x : \alpha \to \alpha, \text{ but } \not\vdash \lambda xy.xy : \alpha \to \alpha.$$

Let us quote some known properties that are concerned with environments.

Proposition 2.9 *(i) If $\Gamma \vdash M : \sigma$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash M : \sigma$.*

(ii) (Barendregt et al. [1983], Lemma 2.8)
If $\Gamma \vdash MN : \sigma$, then there exists a type τ such that $\Gamma \vdash M : \tau \to \sigma$ and $\Gamma \vdash N : \tau$.

The notions of *principal type* and *principal typing* of a normal form are introduced in Coppo et al. [1980] and Ronchi della Rocca et al. [1984], and can be found in Krivine [1990]. The most important property of the principal type of a normal form N is that all other types that type N can be obtained from the principal type by some operations on types. The procedure for obtaining the principal type of a normal form will not be described in detail here, since we will be dealing with some other properties of the principal type.

Definition 2.10 *A* principal typing $\Gamma \vdash N : \sigma$ *of the normal form* N *is defined by cases as follows:*

(i) for $N \equiv x_i$*, then* $x_1 : \alpha_1, \ldots, x_k : \alpha_k \vdash x_i : \alpha_i$ *is a principal typing of* N*, where* $\alpha_1, \ldots, \alpha_k$ *are distinct atomic types,*

(ii) for $N \equiv \lambda x.P$*, then* $x_1 : \varphi_1, \ldots, x_k : \varphi_k \vdash \lambda x.P : \varphi \to \psi$ *is a principal typing of* N*, where* $x : \varphi, x_1 : \varphi_1, \ldots, x_k : \varphi_k \vdash P : \psi$ *is a principal typing of* P*,*

(iii) for $N \equiv xN_1 \ldots N_m$*, then*

$$x : \bigcap_{i=1}^{m} \varphi_i \cap (\psi_1 \to \ldots \to \psi_m \to \alpha), x_1 : \bigcap_{i=1}^{m} \varphi_1^i, \ldots, x_k : \bigcap_{i=1}^{m} \varphi_k^i \vdash xN_1 \ldots N_m : \alpha$$

is a principal typing of N*, where* $x : \varphi_i, x_1 : \varphi_1^i, \ldots, x_k : \varphi_k^i \vdash N_i : \psi_i$ *are principal typings of* $N_i, 1 \leq i \leq m$*, such that for all* $i \neq j$ *the typings of* N_i *and* N_j *have no joint atomic types and* α *is a new atom, fresh for all typings.*

We will be dealing with the following property of principal typings.

Proposition 2.11 *(i) Let* N *be a normal form, and let* $\Gamma \vdash N : \pi$ *be a principal typing of* N*. Then,* $\Gamma \vdash M : \pi$ *implies that there is an* η*-reduct* Q *of* N*,* $N \twoheadrightarrow_\eta Q$*, such that* $M \twoheadrightarrow_\beta Q$*.*

(ii) Let N *be a* $\beta\eta$*-normal form, and let* $\Gamma \vdash N : \pi$ *be a principal typing of* N*. Then,* $\Gamma \vdash M : \pi$ *implies that* $M \twoheadrightarrow_\beta N$*.*

Proof. (i) The proof is given in Krivine [1990].
(ii) Straightforward by (i). □

3. Topologies on Λ

Let us introduce topologies on the set Λ of all lambda terms via typeability of lambda terms in $\lambda\cap$. Let us consider sets of lambda terms that can be typeable by a given type in a given environment:

$$\mathcal{V}_{\Gamma,\sigma} = \{M \mid \Gamma \vdash M : \sigma\}.$$

Lemma 3.1 *(i) For every fixed environment* $\Gamma \in \mathbf{E}$*,* $\mathcal{V}_{\Gamma,\omega} = \Lambda$*.*

(ii) For every fixed environment $\Gamma \in \mathbf{E}$ *and for any types* σ *and* τ *we have that* $\mathcal{V}_{\Gamma,\sigma} \cap \mathcal{V}_{\Gamma,\tau} = \mathcal{V}_{\Gamma,\sigma\cap\tau}$*.*

Proof.

(i) Every lambda term M is typeable by ω in any environment Γ, i.e. $\Gamma \vdash M : \omega$, which means that $\mathcal{V}_{\Gamma,\omega} = \Lambda$ for every Γ.

(ii) If $M \in \mathcal{V}_{\Gamma,\sigma} \cap \mathcal{V}_{\Gamma,\tau}$, then $\Gamma \vdash M : \sigma$ and $\Gamma \vdash M : \tau$. By $(\cap I)$ we have that $\Gamma \vdash M : \sigma \cap \tau$. Therefore $M \in \mathcal{V}_{\Gamma,\sigma\cap\tau}$.

On the other hand $M \in \mathcal{V}_{\Gamma,\sigma\cap\tau}$ means that $\Gamma \vdash M : \sigma \cap \tau$. By $(\cap E)$ we have that $\Gamma \vdash M : \sigma$ and $\Gamma \vdash M : \tau$. Thus, $M \in \mathcal{V}_{\Gamma,\sigma} \cap \mathcal{V}_{\Gamma,\tau}$. $\Box$

Proposition 3.2 *For a fixed* $\Gamma \in \mathbf{E}$ *the family of sets* $\{\mathcal{V}_{\Gamma,\sigma}\}_{\sigma\in\mathtt{type}}$ *forms the basis of a topology on* Λ.

Proof. Immediate by Lemma 3.1 and the definition of basis. $\Box$

This topology will be called the Γ-*fit topology. Open sets* in the Γ-fit topology are defined in the usual way as unions of basic open sets. Let us notice that each of these topologies depends on the environment Γ and therefore for each environment $\Gamma \in \mathbf{E}$ there is a corresponding Γ-fit topology.

In the sequel we give short and easy proofs of some fundamental theorems of the untyped lambda calculus such as the Continuity Theorem and the Genericity Lemma, which are proved in Barendregt [1984] (Theorem 14.3.22 and Proposition 14.3.24) using the tree topology.

Theorem 3.3 (Continuity Theorem)
The map $App : \Lambda \times \Lambda \to \Lambda$ *defined by* $App(F, M) = FM$ *is continuous with respect to the each* Γ-*fit topology.*

Proof.
We have to show that for a fixed $\Gamma \in \mathbf{E}$ and for a given $\mathcal{V}_{\Gamma,\varepsilon} \ni FM$ there exists a $\mathcal{V}_{\Gamma,\delta_1}$ and a $\mathcal{V}_{\Gamma,\delta_2}$ such that $F \in \mathcal{V}_{\Gamma,\delta_1}$ and $M \in \mathcal{V}_{\Gamma,\delta_2}$, and for any $G \in \mathcal{V}_{\Gamma,\delta_1}$ and $N \in \mathcal{V}_{\Gamma,\delta_2}$ we have $GN \in \mathcal{V}_{\Gamma,\varepsilon}$, i.e.

$$(\forall \mathcal{V}_{\Gamma,\varepsilon} \ni FM)(\exists \mathcal{V}_{\Gamma,\delta_1} \ni F)(\exists \mathcal{V}_{\Gamma,\delta_2} \ni M)(G \in \mathcal{V}_{\Gamma,\delta_1} \wedge N \in \mathcal{V}_{\Gamma,\delta_2} \Rightarrow GN \in \mathcal{V}_{\Gamma,\varepsilon}).$$

If $FM \in \mathcal{V}_{\Gamma,\varepsilon}$, then $\Gamma \vdash FM : \varepsilon$. By the structural property given in Proposition 2.9 (ii) there is a type δ such that

$$\Gamma \vdash F : \delta \to \varepsilon \text{ and } \Gamma \vdash M : \delta.$$

Taking $\delta_1 \equiv \delta \to \varepsilon$ and $\delta_2 \equiv \delta$ we prove the statement. $\Box$

Let us consider some properties of the Γ-fit topology that are related to β- and η-reduction.

Proposition 3.4 *Let $M, P \in \Lambda$.*

(i) If $M \twoheadrightarrow_\beta P$, then $\forall \Gamma \in \mathbf{E}\ \forall \sigma \in \mathtt{type}\ \ (M \in \mathcal{V}_{\Gamma,\sigma} \Leftrightarrow P \in \mathcal{V}_{\Gamma,\sigma})$.

(ii) If $M \twoheadrightarrow_\eta P$, then $\forall \Gamma \in \mathbf{E}\ \forall \sigma \in \mathtt{type}\ \ (M \in \mathcal{V}_{\Gamma,\sigma} \Rightarrow P \in \mathcal{V}_{\Gamma,\sigma})$.

(iii) If $N \in \Lambda$ is a $\beta\eta$–normal form, then

$$M \twoheadrightarrow_\beta N \text{ if and only if } \forall \Gamma \in \mathbf{E}\ \forall \sigma \in \mathtt{type}\ \ (M \in \mathcal{V}_{\Gamma,\sigma} \Leftrightarrow N \in \mathcal{V}_{\Gamma,\sigma}).$$

Proof.

(i) This is a consequence of the subject reduction and expansion property for β-reduction which holds for $\lambda\cap$ (Theorem 2.8(i)).

(ii) This is a consequence of the subject reduction property for η-reduction which holds for $\lambda\cap$ (Theorem 2.8(ii)).

(iii) ($\Rightarrow$) By (i).
($\Leftarrow$) Let $\vdash N : \pi$ be a principal typing of N. Then by the assumption $\vdash M : \pi$. Therefore, by Proposition 2.11(ii) $M \twoheadrightarrow_\beta N$. □

Lemma 3.5 *(i) $\mathcal{V}_{\Gamma,\omega} = \Lambda$ for all $\Gamma \in \mathbf{E}$.*

(ii) If $\sigma \sim \tau$, then $\mathcal{V}_{\Gamma,\sigma} = \mathcal{V}_{\Gamma,\tau}$ for all $\Gamma \in \mathbf{E}$.

Proof. (i) Obvious, since every lambda term is trivially typeable by ω.
(ii) If $M \in \mathcal{V}_{\Gamma,\sigma}$ and $\sigma \sim \tau$, then $M \in \mathcal{V}_{\Gamma,\tau}$, and vice versa. □

We show that if M is an unsolvable term, then the only open set containing M is Λ itself, thus the unsolvable terms are bottoms (generic points). Also, we show that $\beta\eta$-normal forms are isolated points, i.e. that there are open sets containing them which are singletons.

Proposition 3.6 *(i) Unsolvable terms are bottoms (generic points) with respect to each Γ-fit topology.*

(ii) If N is a normal form, then there is a Γ-fit topology, with the basis $\{\mathcal{V}_{\Gamma,\sigma}\}_{\sigma\in\mathtt{type}}$, and a type $\rho \in \mathtt{type}$ such that $\mathcal{V}_{\Gamma,\rho} = \{M \mid \exists P \in \Lambda \text{ such that } N \twoheadrightarrow_\eta P \text{ and } M =_\beta P\}$.

(iii) If N is a $\beta\eta$-normal form, then there is a Γ-fit topology and a type $\rho \in \mathtt{type}$ such that $\mathcal{V}_{\Gamma,\rho} = \{M \mid M =_\beta N\}$.

Proof.

(i) Let M be an unsolvable term, then by Theorem 2.7(ii) if $M \in \mathcal{V}_{\Gamma,\sigma}$, then $\sigma \sim \omega$. Therefore, $\mathcal{V}_{\Gamma,\sigma} = \mathcal{V}_{\Gamma,\omega} = \Lambda$ by Lemma 3.5 (i) and (ii). Thus, Λ is the only open set containing unsolvable terms, and hence they are bottoms.

(ii) If N is a normal form, then there is a principal typing $\Gamma \vdash N : \pi$. For every term M for which $\Gamma \vdash M : \pi$, i.e., $M \in \mathcal{V}_{\Gamma,\pi}$ by Proposition 2.11(i) there is an η-reduct P of N such that $M \twoheadrightarrow_\beta P$. According to Proposition 3.4 (ii), $P \in \mathcal{V}_{\Gamma,\pi}$. By Proposition 2.2, P is a normal form as well. Hence, $\mathcal{V}_{\Gamma,\pi} = \{M \mid \exists P \in \Lambda \text{ such that } N \twoheadrightarrow_\eta P \text{ and } M =_\beta P\}$.

(iii) If N is a $\beta\eta$-normal form, then by (ii) there exists a Γ-fit topology $\{\mathcal{V}_{\Gamma,\sigma}\}_{\sigma \in \mathtt{type}}$ and in this case $\mathcal{V}_{\Gamma,\pi} = \{M \mid M =_\beta N\}$. $\square$

Now we can prove the Genericity Lemma (see Barendregt [1984] Proposition 14.3.24) for all lambda terms as a consequence of the Continuity Theorem.

Proposition 3.7 (Genericity Lemma for lambda terms)
Let M be an unsolvable lambda term and let F be a lambda term for which

$$FM =_\beta N,$$

where N has a normal form. Then, for all lambda terms L,

$$FL =_{\beta\eta} N.$$

Proof. Let N_{nf} be the normal form of N, i.e., $N \twoheadrightarrow_\beta N_{nf}$. If $\Gamma \vdash N_{nf} : \pi$ is a principal typing of N_{nf}, then we will consider the Γ-fit topology with the basis $\{\mathcal{V}_{\Gamma,\sigma}\}_{\sigma \in \mathtt{type}}$. According to Proposition 3.6 (ii), $\mathcal{V}_{\Gamma,\pi} = \{Q \mid \exists P \in \Lambda \text{ such that } N_{nf} \twoheadrightarrow_\eta P \text{ and } Q =_\beta P\}$. By Proposition 3.4 (i) $FM, N \in \mathcal{V}_{\Gamma,\pi}$. By the Continuity Theorem 3.3

$$(\forall \mathcal{V}_{\Gamma,\varepsilon} \ni N)(\exists \mathcal{V}_{\Gamma,\delta} \ni M)(L \in \mathcal{V}_{\Gamma,\delta} \Rightarrow FL \in \mathcal{V}_{\Gamma,\varepsilon}).$$

Since $N \in \mathcal{V}_{\Gamma,\pi}$ we can choose $\mathcal{V}_{\Gamma,\varepsilon} = \mathcal{V}_{\Gamma,\pi}$. By Proposition 3.6 (i) $\mathcal{V}_{\Gamma,\omega} = \Lambda$ is the only open set containing the unsolvable term M and hence, $\mathcal{V}_{\Gamma,\delta} = \mathcal{V}_{\Gamma,\omega} = \Lambda$,. Therefore if $L \in \Lambda$, then $FL \in \mathcal{V}_{\Gamma,\pi}$. By Proposition 3.6 (ii) this means that $N_{nf} \twoheadrightarrow_\eta P$ and $FL =_\beta P$, for some $P \in \Lambda$, and so $FL =_{\beta\eta} N$. $\square$

A natural question to be asked is:

How to restrict all Γ-fit topologies to the set Λ^0 of closed lambda terms?.

There is a unique way to do that, since all the Γ-fit topologies collapse into one topology. Let us consider the set of all closed lambda terms that can be typed by a given type, say

$$\mathcal{V}_\sigma = \{M \in \Lambda^0 \mid \vdash M : \sigma\}.$$

Lemma 3.8 *(i) For every closed lambda term M, $M \in \mathcal{V}_\omega$.*

(ii) For any types σ and τ, we have that $\mathcal{V}_\sigma \cap \mathcal{V}_\tau = \mathcal{V}_{\sigma\cap\tau}$.

Proof. (i) Every closed lambda term M is typeable by ω in $\lambda\cap$, i.e. $\vdash M : \omega$, which means $M \in \mathcal{V}_\omega$.
(ii) Similar to Lemma 3.1 (ii). $\Box$

Proposition 3.9 *The family $\{\mathcal{V}_\sigma\}_{\sigma\in\mathtt{type}}$ forms a basis for a topology on Λ^0.*

Proof. Immediate by Lemma 3.8 and the definition of basis. $\Box$

This topology will be called the *fit topology*. *Open sets* in the fit topology are defined in the usual way as unions of basic open sets.

Proposition 3.10 *For all $\Gamma \in \mathbf{E}$ and all $\sigma \in \mathtt{type}$ we have that $\mathcal{V}_\sigma \subseteq \mathcal{V}_{\Gamma,\sigma}$.*

Proof. If $M \in \mathcal{V}_\sigma$ which means that $\vdash M : \sigma$. Then, for all $\Gamma \in \mathbf{E}$, according to Proposition 2.9 (i), $\Gamma \vdash M : \sigma$ which means that $M \in \mathcal{V}_{\Gamma,\sigma}$. $\Box$

Results analogous to those in this section (3.3–3.7) hold in Λ^0 as well. The proofs of the Continuity Theorem and the Genericity Lemma for closed lambda terms are straightforward by Theorem 3.3 and Proposition 3.7: we leave out the environment Γ from the corresponding proof.

As soon as we have topologized Λ, we get the *specialization preorder* (see Vickers [1989], Chapter 7) defined as follows.

Definition 3.11 *(i) $M \sqsubseteq N$ iff every open set containing M also contains N (it is sufficient to consider only basic open sets).*

(ii) $M \sim N$ iff $M \sqsubseteq N$ and $N \sqsubseteq M$ (the corresponding equivalence relation).

The following statement is a consequence of Proposition 3.4 (i) and (ii).

Proposition 3.12 *(i) If $M \twoheadrightarrow_\beta N$, then $M \sim N$.*

(ii) If $M \twoheadrightarrow_\eta N$, then $M \sqsubseteq N$.

The equivalence $\sim$ cannot be completely described in terms of $\twoheadrightarrow_\beta$. The following statement gives a partial converse of Proposition 3.12 (i).

Proposition 3.13 *(i) Let N be a $\beta\eta$–normal form. $M \sim N$ iff $M \twoheadrightarrow_\beta N$.*

(ii) Let M and N be normalizing terms. $M \sim N$ iff $M =_\beta N$.

(iii) Let M and N be unsolvable terms. Then $M \sim N$.

Proof. (i) Let N be a $\beta\eta$–normal form and let $M \sim N$. Then $\Gamma \vdash M : \pi$ where $\Gamma \vdash N : \pi$ is a principal typing of N. Then $M \twoheadrightarrow_\beta N$ according to Proposition 2.11 (ii).

(ii) Let M and N be normalizing. The normal form of a lambda term is unique and hence there are normal forms M_{nf} and N_{nf} such that $M \twoheadrightarrow_\beta M_{nf}$ and $N \twoheadrightarrow_\beta N_{nf}$. If $M \sim N$, then $M_{nf} \sim N_{nf}$ which means that they have the same principal types. Suppose $M \not\equiv N$, then by Proposition 2.11 (i) there exists a term M' (N') which is an η– reduct of M_{nf} (N_{nf}) such that $N_{nf} \twoheadrightarrow_\beta M'$ ($M_{nf} \twoheadrightarrow_\beta N'$), which is not possible, since N_{nf} and M_{nf} are normal forms. That means that M_{nf} and N_{nf} are syntactically equal, $M_{nf} \equiv N_{nf}$. Therefore $M =_\beta N$.

(iii) This is a consequence of Proposition 3.6 (i), which states that unsolvable terms are bottoms. □

The following statement partially describes $\sqsubseteq$ in terms of reductions giving a partial converse of Proposition 3.12. (ii).

Proposition 3.14 (Characterization of $\sqsubseteq$)

(i) Let M be a normalizing term, with normal form M_{nf}. Then $M \sqsubseteq N$ iff for some closed term P both $M_{nf} \twoheadrightarrow_\eta P$ and $N \twoheadrightarrow_\beta P$.

(ii) Let M be an unsolvable. Then for every $N \in \Lambda$ we have that $M \sqsubseteq N$.

Proof. (i) This is a direct consequence of Proposition 2.11 (i).
(ii) This is again a consequence of the fact that unsolvable terms are bottoms (Proposition 3.6 (i)). □

4. Comparing topologies

In this section we compare the filter topology on the set of lambda terms and the fit topology. We show on the one hand that the fit topology coincides with the filter topology on Λ^0, and on the other hand that for each Γ-fit topology on Λ there is a coincident filter topology on Λ and the other way round. It appears that the fit topology is a simpler description of the filter topology since the main difference between these topologies is that the former is a topology introduced on the set of types and then traced on terms by the inverse map, whereas the latter is introduced directly on the set of terms. First, let us recall some basic notions and properties of the filter topology.

Filter models are introduced in Barendregt et al. [1983] in order to show the completeness of the type assignment in $\lambda\cap$. The basic idea of filter models is that lambda models can be obtained from the set `type` of intersection types with the partial ordering $\leq$.

Definition 4.1 *A filter is a subset $d \subseteq$ `type` such that:*

(i) $\omega \in d$;

(ii) $\sigma, \tau \in d \Rightarrow \sigma \cap \tau \in d$;

(iii) $\sigma \in d$, $\sigma \leq \tau \Rightarrow \tau \in d$.

Let $\mathcal{F}$ denote the set of all filters.

Lemma 4.2 *(Barendregt et al. [1983])* $\{\sigma \mid \Gamma \vdash M : \sigma\}$ *is a filter.*

$\mathcal{F}$ is a complete lattice under set theoretical inclusion and can be provided with the Scott topology in a standard way.

Definition 4.3 *A set* $\mathcal{O} \subseteq \mathcal{F}$ *is open if:*

(i) $d \in \mathcal{O}$ *and* $d \subseteq e$ *imply that* $e \in \mathcal{O}$;

(ii) $\cup_{i \in I} d_i \in \mathcal{O}$ *implies that there is an* $i_0 \in I$ *such that* $d_{i_0} \in \mathcal{O}$.

Definition 4.4 *(i) The principal filter generated by the type* σ *is the set* $\widehat{\sigma} = \{\tau \mid \sigma \leq \tau\}$.

(ii) The upper closure of a filter d *is the set* $\uparrow d = \{e \mid d \subseteq e\}$.

From the domain theory (see e.g. Vickers [1989]) it follows that a basis for the Scott topology is given by the upper closure of principal filters.

Proposition 4.5 *Let* $\sigma \in \texttt{type}$. *Then the upper closure of the principal filter* $\uparrow \widehat{\sigma}$ *is open in the Scott topology on* $\mathcal{F}$.

Proof. Let us show that $\uparrow \widehat{\sigma} = \{d \mid \widehat{\sigma} \subseteq d\} \subseteq \mathcal{F}$ is open in the filter topology.

(i) Let $d \in \uparrow \widehat{\sigma}$ and $d \subseteq e$. Then $e \in \uparrow \widehat{\sigma}$ by Definition 4.4 (ii) of the upper closure.

(ii) If $\bigcup_{i \in I} d_i \in \uparrow \widehat{\sigma}$, then $\widehat{\sigma} \subseteq \bigcup_{i \in I} d_i$. It means that $\sigma \in \bigcup_{i \in I} d_i$ and then there exists $i_0 \in I$ such that $\sigma \in d_{i_0}$. Thus $\widehat{\sigma} \subseteq d_{i_0}$, i.e., $d_{i_0} \in \uparrow \widehat{\sigma}$. $\square$

Lemma 4.6 *Let* $d \in \mathcal{F}$. *Then* $d = \cup_{\sigma \in d} \widehat{\sigma}$.

Proof. If a type $\rho \in d$, then $\rho \in \widehat{\rho}$ and hence $\rho \in \cup_{\sigma \in d} \widehat{\sigma}$. On the other hand, if $\rho \in \cup_{\sigma \in d} \widehat{\sigma}$, then $\rho \in \widehat{\sigma}_i$ for some $\sigma_i \in d$. Then by Definition 4.4 (i) $\sigma_i \leq \rho$ and hence $\rho \in d$. $\square$

Proposition 4.7 *A basis for the Scott topology on the set* $\mathcal{F}$ *of filters is the set of upper closures of principal filters* $\{\uparrow \widehat{\sigma} \mid \sigma \in \texttt{type}\}$.

Proof. (i) For every $d \in \mathcal{F}$ there is an open set $\mathcal{O}$ in the Scott topology on $\mathcal{F}$ such that $d \in \mathcal{O}$. By the previous Lemma 4.6 $d = \cup_{\sigma \in d} \widehat{\sigma} \in \mathcal{O}$. According to Definition 4.3 (ii) of open sets there is a $\sigma_i \in d$ such that $\widehat{\sigma}_i \in \mathcal{O}$. Therefore $\uparrow \widehat{\sigma}_i \subseteq \mathcal{O}$ and $d \in \uparrow \widehat{\sigma}_i$.

(ii) Let us show that for every $\uparrow \widehat{\sigma}$ and $\uparrow \widehat{\tau}$ from the family $\{\uparrow \widehat{\sigma} \mid \sigma \in \texttt{type}\}$ there is a set from the same family which is contained in their intersection and that set is $\uparrow \widehat{\sigma \cap \tau}$.

Moreover, we show that $\uparrow \widehat{\sigma \cap \tau} = \uparrow \hat{\sigma} \cap \uparrow \hat{\tau}$. Let $d \in \mathcal{F}$.

$$\begin{aligned} d \in \uparrow \widehat{\sigma \cap \tau} &\Leftrightarrow \widehat{\sigma \cap \tau} \subseteq d \\ &\Leftrightarrow \sigma \cap \tau \in d \\ &\Leftrightarrow \sigma \in d \text{ and} \tau \in d \\ &\Leftrightarrow \hat{\sigma} \subseteq d \text{ and } \hat{\tau} \subseteq d \\ &\Leftrightarrow d \in \uparrow \hat{\sigma} \text{ and } d \in \uparrow \hat{\tau} \\ &\Leftrightarrow d \in \uparrow \hat{\sigma} \cap \uparrow \hat{\tau}. \qquad \Box \end{aligned}$$

Let us notice here that the Scott topology on $\mathcal{F}$ and the related general topological results up to now (4.3 –4.7) are discussed without any reference to lambda calculus. In order to bring in lambda calculus we need to get the valuation of (closed) lambda terms in filters ($\| \ \| : \Lambda^0 \to \mathcal{F}$) $\| \ \| : \Lambda \to \mathcal{F}$ that will enable us to pull back the topology from $\mathcal{F}$ to (Λ^0) Λ.

This *valuation of closed lambda terms in filters* $\| \ \| : \Lambda^0 \to \mathcal{F}$ is given by the mapping

$$\|M\| = \{\sigma \mid \ \vdash M : \sigma\}.$$

We have that $\|M\| \in \mathcal{F}$ by Lemma 4.2.

The filter topology on Λ^0 is the smallest that makes $\| \ \| : \Lambda^0 \to \mathcal{F}$ continuous w.r.t. the Scott topology on $\mathcal{F}$. Open sets in the filter topology on Λ^0 are obtained from open sets in the Scott topology on $\mathcal{F}$ by the inverse map, i.e. the Scott topology on $\mathcal{F}$ induces the filter topology on Λ^0 in the sense that the open sets in Λ^0 are $\|\mathcal{O}\|^{-1} \subseteq \Lambda^0$, where $\mathcal{O} \subseteq \mathcal{F}$ is open in the Scott topology on $\mathcal{F}$.

Now, we can compare the fit topology introduced in the previous section and the filter topology. The following proposition shows that these topologies coincide on the set Λ^0 of closed lambda terms, since their bases are the same.

Proposition 4.8 *Let $\sigma \in \mathtt{type}$. Then $\| \uparrow \hat{\sigma}\|^{-1} = \mathcal{V}_\sigma$.*

Proof. Let P be a closed lambda term.

$$\begin{aligned} P \in \| \uparrow \hat{\sigma}\|^{-1} &\Leftrightarrow \|P\| \in \uparrow \hat{\sigma} \\ &\Leftrightarrow \hat{\sigma} \subseteq \|P\| \\ &\Leftrightarrow \sigma \in \|P\| \\ &\Leftrightarrow \vdash P : \sigma \\ &\Leftrightarrow P \in \mathcal{V}_\sigma. \qquad \Box \end{aligned}$$

Therefore the filter topology and the fit topology are the same on the set of closed lambda terms. The advantage of the fit topology is that it is defined on Λ^0 whereas the filter topology is actually induced by the topology defined on $\mathcal{F}$.

Again, a natural question to be asked is:

How to extend the filter topology to the set of all lambda terms?.

A *valuation of lambda terms in filters* $\| \ \|_\Gamma : \Lambda \to \mathcal{F}$ is given by the map

$$\|M\|_\Gamma = \{\sigma \mid \Gamma \vdash M : \sigma\}$$

and it depends on the environment Γ and hence it is not unique. Obviously, $\|M\|_\Gamma$ is a filter according to Lemma 4.2.

The Γ-filter topology on Λ is the smallest that makes the map $\| \ \|_\Gamma$ continuous w.r.t. the Scott topology on $\mathcal{F}$. For every environment Γ there is a Γ-filter topology and therefore there are infinitely many Γ-filter topologies on Λ. Open sets in Λ are obtained from open sets in $\mathcal{F}$ by the inverse map.

Proposition 4.9 *For every* $\Gamma \in \mathbf{E}$ *the family*

$$\{\| \uparrow \widehat{\sigma} \|_\Gamma^{-1}\}_{\sigma \in \mathtt{type}}$$

is a basis for the Γ*-filter topology on* Λ*.*

Proof. This holds since $\{\uparrow \widehat{\sigma}\}_{\sigma \in \mathtt{type}}$ is a basis for the Scott topology on $\mathcal{F}$ by Proposition 4.7. $\square$

In order to compare the Γ-fit and the Γ-filter topologies on Λ we show that the corresponding topologies have the same basis, meaning that each Γ-fit topology has a corresponding Γ-filter topology with the same basis and vice versa.

Proposition 4.10 *Let* $\Gamma \in \mathbf{E}$ *and* $\sigma \in \mathtt{type}$*. Then* $\| \uparrow \widehat{\sigma} \|_\Gamma^{-1} = \mathcal{V}_{\Gamma,\sigma}$*.*

Proof. Let $P \in \Lambda$.

$$\begin{aligned} P \in \| \uparrow \widehat{\sigma} \|_\Gamma^{-1} &\Leftrightarrow \|P\|_\Gamma \in \uparrow \widehat{\sigma} \\ &\Leftrightarrow \widehat{\sigma} \subseteq \|P\|_\Gamma \\ &\Leftrightarrow \sigma \in \|P\|_\Gamma \\ &\Leftrightarrow \Gamma \vdash P : \sigma \\ &\Leftrightarrow P \in \mathcal{V}_{\Gamma,\sigma}. \qquad \square \end{aligned}$$

5 . Discussion

The *tree topology* is used in Barendregt [1984] in order to prove the Continuity Theorem and the Genericity Lemma. It is based on the notion of Böhm trees. The tree topology on the set Λ of all untyped lambda terms is the smallest that makes the mapping $BT : \Lambda \to \mathcal{B}$ continuous, where $\mathcal{B}$ is the set of all Böhm-like trees. This means that the open sets in Λ

are of the form $BT^{-1}(\mathcal{O})$, with $\mathcal{O}$ open w.r.t. the Scott topology on $\mathcal{B}$. The sets $\mathcal{O}_{M,k} = \{N \in \Lambda | M^{(k)} \sqsubseteq N\}$, form a basis w.r.t. this topology on Λ, where $M^{(k)}$ is the lambda term reconstructed from $BT^k(M)$, the Böhm tree of the lambda term M of length k. The set $\mathcal{O}_{M,k}$ consists of all lambda terms whose Böhm trees contain the same Böhm tree of $M^{(k)}$.

β-equal lambda terms are separable neither in the tree topology, since they have equal head normal forms, and hence the same Böhm tree, nor in the type topology, since they are typeable by the same types (Theorem 2.8 (i)). On the other hand η-equal terms can be separated in both of these topologies, since they do not have necessarily the same head normal form and they are not typed by the same types. For example, η-equal terms $\mathbf{1} \equiv \lambda xy.xy \to_\eta \lambda x.x \equiv \mathsf{I}$ are not typeable by the same types. Both of them are head normal forms, even normal forms, but not equal, which means that their Böhm trees are different.

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