The paradoxes and the infinite dazzled ancient mathematics and continue to do so today

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Talk based on:

Fairouz Kamareddine and Jonathan Seldin
"A primer of Mathematical Analysis and the Foundations of Computation"

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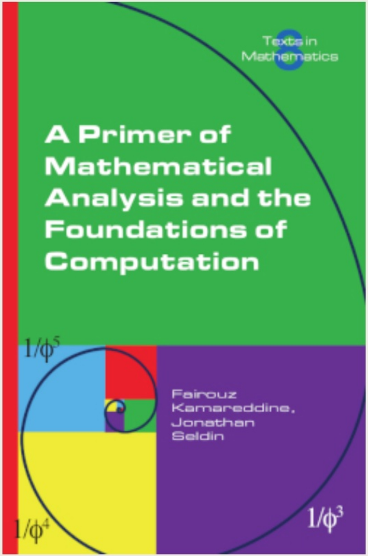


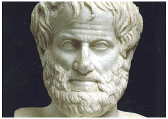
A Primer of Mathematical Analysis and the Foundations of Computation

Fairouz Kamareddine and Jonathan Seldin

This book is a different approach to teaching the foundations of mathematical analysis and of computation. The main idea is to delay the use of "formal definitions", which are definitions that nobody can understand without working with them. The approach of this book is to employ the history of mathematics to first develop fundamental concepts of mathematical analysis and the theory of computation and to only introduce formal definitions after the concepts are understood by the students

The historical order clarifies what analysis is really about and also why the theory of computation came about. The book provides students with a broader background involving for instance glimpses of cardinal arithmetic, predicate logic background, as well as the importance of a sound theory of the infinitesimal (which is in essence the foundations of mathematics and computation).

There is a wealth of exercises (with solutions in a separate booklet available to download from the link below) and numerous graphical illustrations which give an experienced instructor lots of possibilities to select a stimulating course with a broader background. Even for just browsing by general readers, this book presents stories, insights and mathematical theories, covering a window of ancient times to the present.



- Assume a problem Π,
  - If you *give* me an algorithm to solve Π, I can check whether this algorithm really solves Π.
  - But, if you ask me to *find* an algorithm to solve Π, I may go on forever trying but without success.
- But, this result was already known to Aristotle:
- Assume a proposition Φ.
  - If you *give* me a proof of Φ, I can check whether this proof really proves Φ.
  - But, if you ask me to *find* a proof of Φ, I may go on forever trying but without success.
- In fact, *programs* are *proofs*:
  - *program* = *algorithm* = *computable function* = *λ-term*.
  - By the PAT principle: *Proofs* are *λ-terms*.

> If we could find characters or signs appropriate for expressing all our thoughts as definitely and as exactly as arithmetic expresses numbers or geometric analysis expresses lines, we could in all subjects in so far as they are amenable to reasoning accomplish what is done in Arithmetic and Geometry. Leibniz

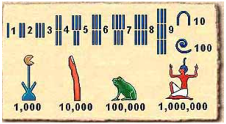
Leibniz (1646–1717) conceived of **automated deduction**, i.e., to find

- a language L in which arbitrary concepts could be formulated, and
- a method to determine the correctness of statements in L.

Leibniz wanted a language and a method that could carry out proof checking and proof finding. However, according to later results by Gödel and Church and Turing, such a method can not work for every statement.

A short history of numbers

- *natural numbers* $(\mathbb{N}^+, =, +, \cdot, 1)$ like 1, 2, which were used to count (using pebbles/stones, strokes, etc.).



- *Integers* like 0, 1, -1, 2, -2, etc.
- In fourteenth century Italy, negative numbers were not known. A double entry bookkeeping system compensated for their absence.
- Accounts in which debits may be greater than credits were compared without using negative integers.
- If c and d are in $\mathbb{N}^+$, then *account* $c \ominus d$ has credit c and debit d.
- Define (accounts, $\cong, +_c, \cdot_c$):
  - *have the same value* $m \ominus n \cong p \ominus q$ iff $m + q = n + p$.
  - $(m \ominus n) +_c (p \ominus q) = (m + p) \ominus (n + q)$ and
    $(m \ominus n) \cdot_c (p \ominus q) = (mp + nq) \ominus (mq + np)$.
  - Equivalence classes: $[m \ominus n] = \{p \ominus q : p \ominus q \cong m \ominus n\}$.
- The set of Integers: $\mathbb{Z} = \{[m \ominus n] | m, n \in \mathbb{N}^+\}$.
  $[(m \ominus n)] +_i [(p \ominus q)] = [(m \ominus n) +_c (p \ominus q)]$ and
  $[(m \ominus n)] \cdot_i [(p \ominus q)] = [(m \ominus n) \cdot_c (p \ominus q)]$.
- Identity for $+_i$: For any m, n in $\mathbb{N}^+$, $[m \ominus m] = [n \ominus n]$. Call it 0.
  Identity for $\cdot_i$: Similarly, let $1_i = [(p + 1) \ominus p]$ for any p in $\mathbb{N}^+$.
  Inverse for $+_i$: If $\alpha = [m \ominus n]$, then $-\alpha = [n \ominus m]$.
- $(\mathbb{Z}, +_i, \cdot_i, 0, 1_i, -\alpha)$.

- *Rational numbers* which are the values of fractions of integers like $2/3$.
- $(\text{fractions}, \asymp, +_f, \cdot_f)$: Let $m, n, p, q \in \mathbb{N}^+$.
  - $\frac{m}{n} \asymp \frac{p}{q}$ if and only if $mq = np$,
  - $\frac{m}{n} +_f \frac{p}{q} = \frac{mq + np}{nq}$ and $\frac{m}{n} \cdot_f \frac{p}{q} = \frac{mp}{nq}$.
  - Equivalence class: $\left[\frac{m}{n}\right] = \left\{\frac{p}{q} : \frac{p}{q} \asymp \frac{m}{n}\right\}$.
  - Set of positive rational numbers: $\mathbb{Q}^+ = \{\left[\frac{m}{n}\right] | m, n \in \mathbb{N}^+\}$.
- $\left[\frac{m}{n}\right] +_r \left[\frac{p}{q}\right] = \left[\frac{m}{n} +_f \frac{p}{q}\right]$.
- $\left[\frac{m}{n}\right] \cdot_r \left[\frac{p}{q}\right] = \left[\frac{m}{n} \cdot_f \frac{p}{q}\right]$.
- $(\mathbb{Q}^+, +_r, \cdot_r, 1_r, \mathbf{a}^{-1})$ where $1_r = \left[\frac{1}{1}\right]$ and $\left[\frac{m}{n}\right]^{-1} = \left[\frac{n}{m}\right]$

	$\mathbb{N}^+$	$\mathbb{Q}^+$	$\mathbb{Z}$
Equivalence relations		$\asymp$	$\cong$
Definitions of sets		equiv. classes of fractions	equiv. classes of accounts
Operations	$+, \cdot$	$+_r, \cdot_r$	$+_i, \cdot_i,$
Closure of operations	$\surd$	$\surd$	$\surd$
Commutativity of operations	$\surd$	$\surd$	$\surd$
Associativity of operations	$\surd$	$\surd$	$\surd$
Distributivity of multiplication over addition	$\surd$	$\surd$	$\surd$
$\mathbb{N}^+ \subseteq$ set		$\surd$	$\surd$
Cancellation of operations	$\surd$	$\surd$	$\surd$
Identity for addition	$\times$	$\times$	$\surd$ 0
Identity for multiplication	$\surd$ 1	$\surd$ 1_r	$\surd$ 1_i
Inverse for addition	$\times$	$\times$	$\surd$
Inverse for multiplication	$\times$	$\surd$	$\times$

Adding Identity Elements and Inverses

- Comparing the theory of fractions and the theory of accounts suggests that we can define a unified theory for adding inverses and, if none is present, identity elements.

$(S, \circ)$ **a Commutative Cancellation Semigroup** (i.e., $\circ$ satisfies closure, commutativity, associativity and cancellation Law on S).

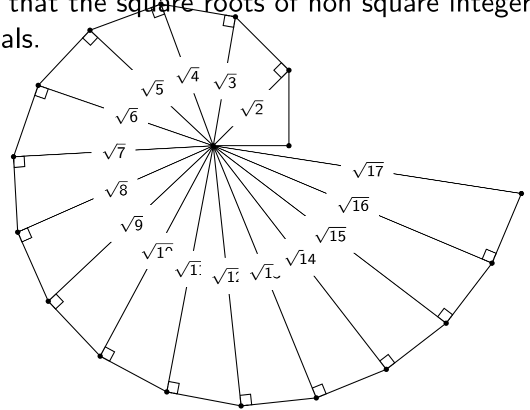
1. **Define congruence $\approx$ on $S \times S$ based on** $(S, \circ)$ by: $(x, y) \approx (u, v)$ iff $x \circ v = y \circ u$.
2. **The operation $*$ on $S \times S$ inherited from** $\circ$ is defined by $(x, y) * (u, v) = (x \circ u, y \circ v)$.
3.
   - Define $[(x, y)] = \{(u, v) : (u, v) \approx (x, y)\}$ and $S_d = \{[(x, y)] : x, y \in S\}$.
   - If $\mathfrak{a} = [(x, y)]$ and $\mathfrak{b} = [(u, v)]$, define $\mathfrak{a} \circ_d \mathfrak{b} = [(x, y) * (u, v)] = [(x \circ u, y \circ v)]$.
   - $(S_d, \circ_d)$ **is a commutative cancellation semigroup**.
   - S **is a subset of** S_d: If $x \in S$, then $x_d = [(y \circ x, y)] \in S_d$.
   - **Identity for Dyads**. Define e_d to be $[(x, x)]$ for some x in S. For all dyads $\mathfrak{a}$, we have $e_d \circ_d \mathfrak{a} = \mathfrak{a} \circ_d e_d = \mathfrak{a}$.
   - **Inverses for Dyads**. If $\mathfrak{a} = [(x, y)]$, define $\mathfrak{a}^{-1}$ to be $[(y, x)]$. We have $\mathfrak{a} \circ_d \mathfrak{a}^{-1} = e_d = \mathfrak{a}^{-1} \circ_d \mathfrak{a}$.

commutative cancellation semigroup	$(\mathbb{N}^+, +)$	$(\mathbb{N}^+, \cdot)$
inverses	$\times$	$\times$
Identity element	$\times$	$\surd$
commutative cancellation semigroup with identity and inverses	$(\mathbb{Z}, +_i)$ $\surd$	$(\mathbb{Q}^+, \cdot_r)$ $\surd$

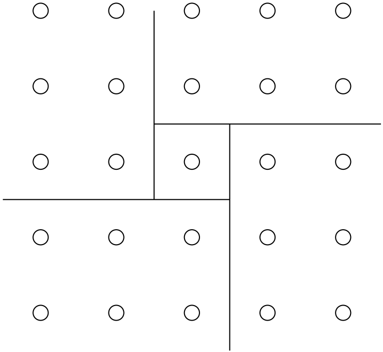
- You can also build $\mathbb{Q}$ this way. But you cannot build $\mathbb{R}$ this way.
- The real numbers need to be constructed (using approximations and limits like Dedekind cuts, Cauchy sequences, etc.)
- This brings us to what is the foundations of mathematics?
- The foundation of mathematics is reasoning about whether the infinitesimal is sound.

Greek mathematics

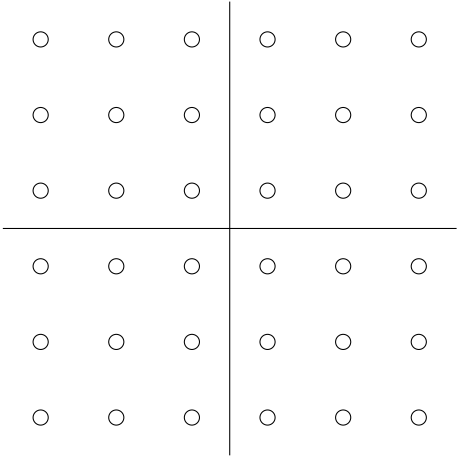
- Euclid's Elements developed mathematics in geometric terms and anything not expressible in such terms was excluded.
- Geometry could accommodate the whole numbers and their ratios as well as irrational magnitudes.
- As an example, take the spiral of Theodorus of Cyrene which established that the square roots of non square integers from 3 to 17 are irrationals.

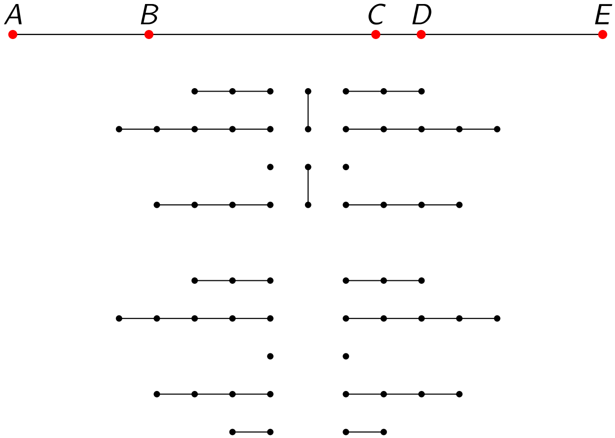


- Knorr suggests that the original proofs were proofs as diagrams using *pebble diagrams*.
- It is known that the ancient Greeks did arithmetic by counting with pebbles, and pebble diagrams give these calculations by representing the pebbles by using small circles.
- The square of every odd number is one more than a multiple of 4.



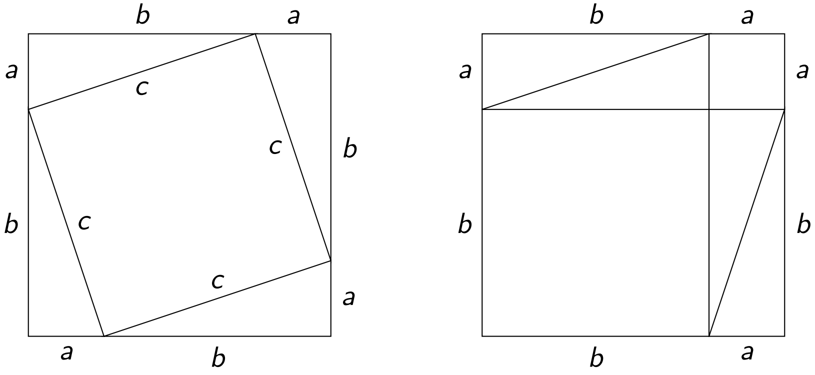
- The square of every even number is a multiple of 4.



- *If as many odd numbers as we please be added together, and their multitude be even, then the sum is even.*
- 

A kind of proof of the Pythagorean Theorem

The Pythagorean Theorem: $c^2 = a^2 + b^2$.



$$(a+b)^2 = 2ab + c^2 \qquad (a+b)^2 = 2ab + a^2 + b^2.$$

Hence, $2ab + c^2 = 2ab + a^2 + b^2$ and $c^2 = a^2 + b^2$.

Theory of Odd and Even Numbers

- *Pythagorean triples* are triples of positive whole numbers representing the lengths of two legs and the hypotenuse of a right triangle.
- I.e., a Pythagorean triple is a triple of positive integers (a, b, c) if and only if $a^2 + b^2 = c^2$.
- E.g. (3, 4, 5), (6, 8, 10), (5, 12, 13), (9, 12, 15), (8, 15, 17)..
- The key results needed for the proof of the incommensurability of the side and diagonal of a square can be proved from diagrams.

> There is no unit which measures exactly the side and diagonal of a square.

- According to Knorr:

> The change from proofs using diagrams/pebbles to proofs as sequences of statements occurred with the discovery of the incommensurability of the side and diagonal of a square.

Diagrams/Pebbles proofs up to incommensurability

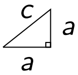
Here are the results you need to prove incommensurability. All can be shown using diagrams/pebbles.

1. In a Pythagorean triple (a, b, c), if c is even, then both a and b are even.
2. In a Pythagorean triple (a, b, c), if c is even, then (a2 , 2b , 2c) is also a Pythagorean triple.
3. In a Pythagorean triple (a, b, c), if c is a multiple of four, so are a and b.
4. In a Pythagorean triple (a, b, c), if c is odd, then one of a and b is odd and the other is even.
5. In a Pythagorean triple (a, b, c), if any two of the numbers is even, the third is also even.
6. In a Pythagorean triple (a, b, c), if one of the numbers is odd, then two of them are odd and one is even.

Incommensurability needs a proof by contradiction

> There is no unit which measures exactly the side and diagonal of a square.

Proof. Suppose there is such a unit in terms of which, the side of the square is a and the diagonal is c.

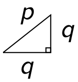
Then, we have a right triangle 

and so (a, a, c) is a Pythagorean triple. Now c must either be even or odd.

- Suppose c even. Then, by 1., a is even. So by 2., we can double the unit and halve all the dimensions. Clearly, we cannot do this indefinitely, since otherwise the unit will grow larger than a.
- So we must have a Pythagorean triple of the form (a, a, c) in which c is odd. But then, by 4., a is both even and odd, a contradiction.

The first proof by contradiction in history?

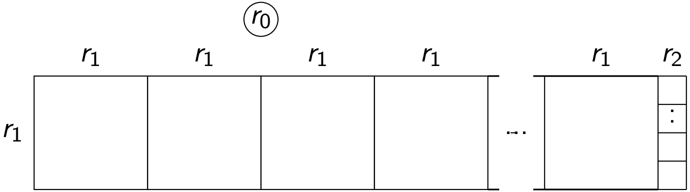
- The proof of the incommensurability theorem is believed to be the first proof by contradiction in the history of mathematiocal proofs.
- The proof cannot be "seen" by looking at a diagram: it is necessary to follow a sequence of sentences with reasons.
- Incommensurability implies that $\sqrt{2}$ is not a rational number.
  Proof:
  - Assume $\sqrt{2} = \frac{p}{q}$, then $2q^2 = p^2$.



  - Hence (q, q, p) forms a Pythagorean triple.
  - Hence there is a unit which measures exactly the side and diagonal of a square.
  - This contradicts the incommensurability theorem.
- The notion of "number" (whole or rational) was no longer enough.
- Discrete collection of units (e.g., naturals or rationals) are not enough.
- We need numbers that are continuous.
- The Greeks did not know how to handle these quantities.

- The main problem was that the Greeks treated mathematical objects as given and did not conceive of constructing them as we did for example when we constructed the positive rationals $\mathbb{Q}^+$ using equivalence classes.
- The ancient Greeks juggled with two notions:
  - Their notion of "numbers" (as a multitude of units, Definition 2 of Book VII).
  - The so-called magnitudes (which in addition to "numbers" include things like lines and areas and volumes, etc.).
  - They developed arithmetic for their numbers, but treated their magnitudes geometrically.
- Starting in the 16th century, in order to construct magnitudes (e.g., the real numbers), approximations were used.
- Even though the Greeks have not thought of constructing new mathematical objects, they did introduce a procedure for approximating ratios.

Euclid used anthyphairesis to find the greatest common divisor of two numbers and to check whether two numbers are prime to one another



- Anthyphairesis is composed of two Greek terms: $\upsilon\phi\alpha\iota\rho\epsilon\omega$ (meaning *subtract*) and $\alpha\nu\tau\iota$ (meaning *alternating/reciprocal*) and hence $\alpha\nu\theta\upsilon\phi\alpha\iota\rho\epsilon\sigma\iota\varsigma$ may stand for *alternated/reciprocal subtraction*.
- Euclid proves that anthyphairesis applied to two relatively prime numbers leads to the unit, and applied to two non relatively prime numbers gives the greatest common divisor of these two numbers.

17 and 3 are prime to one another

PROPOSITION 1. OF BOOK VII OF THE *Elements*
Two unequal numbers being set out, and the less being continuously subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another.

- Take 17 and 3. Then:
  17-3 = 14, 14-3 =11, 11-3 = 8, 8-3 = 5, 5-3 =2, 3-2 =1.
- 14 never measures 17, 11 never measures 14, 8 never measures 11, 5 never measures 8, 2 never measures 5 and we are left with 1.
  - $17 = 5 \times 3 + 2$ where $2 < 3$
  - $3 = 1 \times 2 + 1$ where $1 < 2$
  - $2 = 2 \times 1 + 0$ where $0 < 1$.
  - The ratio is $[5, 1, 2]$ and the continued fraction: $\frac{17}{3} = \boxed{5} + \cfrac{1}{\boxed{1} + \cfrac{1}{\boxed{2}}}$.

Greatest Common Divisor of 136 and 6

Proposition 2. of Book VII of the *Elements*
Given two numbers not prime to one another, to find their greatest common measure.

-

	136	=	22	×	6	+	4.
	6	=	1	×	4	+	2.
	4	=	2	×	2	+	0.

- So, 136 and 6 are not prime to one another and that their greatest common divider is 2.
- The ratio is $[22, 1, 2]$ and the continued fraction is:
$\frac{136}{6} = \boxed{22} + \cfrac{1}{\boxed{1} + \cfrac{1}{\boxed{2}}}$.

Ratio of 12 to 5, [2,2,2]

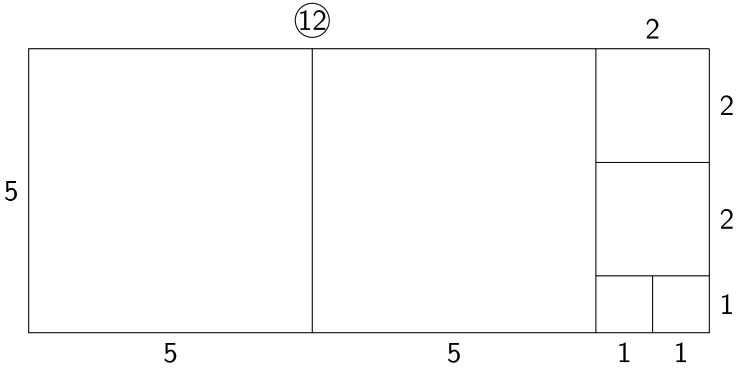


Figure 1: Ratio of 12 to 5

$$\frac{12}{5} = \boxed{2} + \cfrac{1}{\boxed{2} + \cfrac{1}{\boxed{2}}}.$$

Ratio of 22 to 6, [3, 1, 2]

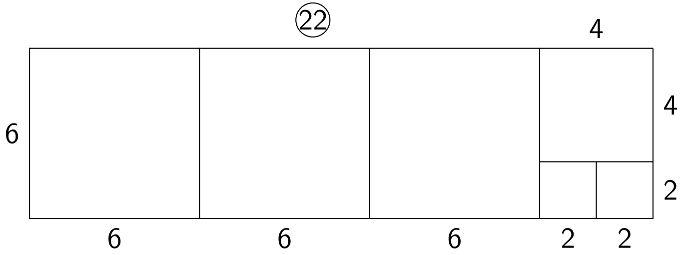


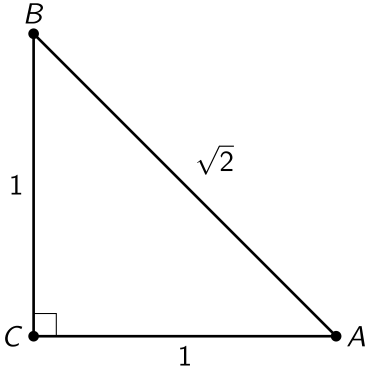
Figure 2: Ratio of 22 to 6

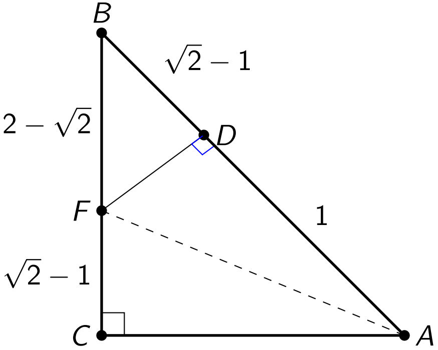
$$\frac{22}{6} = \boxed{3} + \cfrac{1}{\boxed{1} + \cfrac{1}{\boxed{2}}}.$$

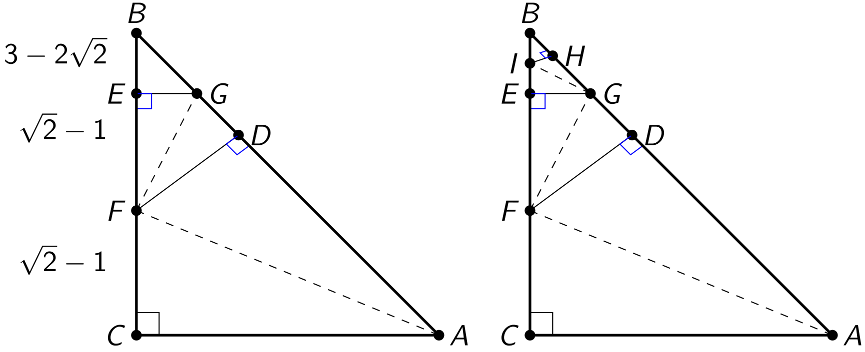
What about Magitudes?

PROPOSITION 2 OF BOOK X OF THE *Elements*.

If, when the less of two unequal magnitudes is continuously subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable.



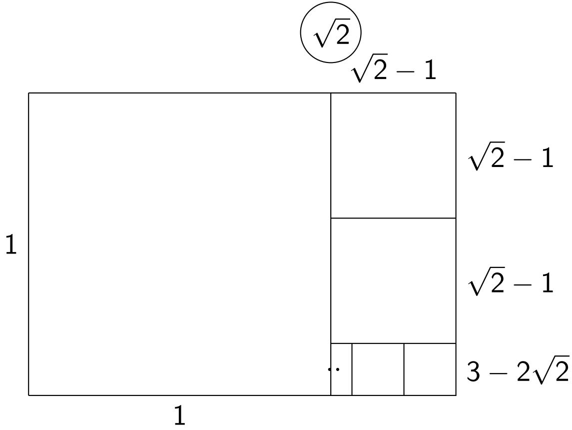




We repeat this process for the isosceles rectangular triangle *BEG* and for the new isosceles rectangular triangle *BIH* and so on. This process can be repeated infinitely.

In this repetition, you see that the less of two unequal magnitudes is continuously subtracted in turn from the greater, yet what is left never measures the one before it.

Ratio of $\sqrt{2}$ to 1, [1,2,2,..]



$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}.$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}.$$

$\sqrt{2}$ is called a quadratic irrational because it is the solution to the quadratic equation $x^2 - 2 = 0$. Note that these continued fractions provide an approximation to $\sqrt{2}$ as follows:

- $\sqrt{2} \approx 1$,
- $\sqrt{2} \approx 1 + \frac{1}{2} = 1.5$,
- $\sqrt{2} \approx 1 + \cfrac{1}{2 + \cfrac{1}{2}} = 1.4$,
- $\sqrt{2} \approx 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = 1.417$,
- $\sqrt{2} \approx 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}} = 1.4139$ etc.

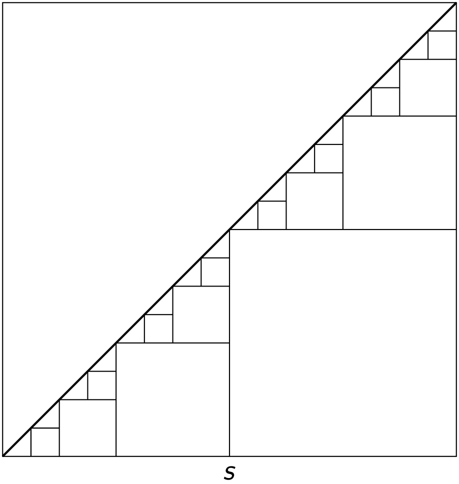
- Two magnitudes are commensurable if and only if anthyphairesis terminates.
- If the anthyphairesis procedure of finding the ratio or GCD of two numbers is applied to incommensurable magnitudes, it will never terminate. We never reach a remainder equal 0.
- Problems with anthyphairesis: some obvious theorems cannot be proved with it:

> *If the ratio of A to C is the same as the ratio of B to C, then $A = B$.*

- Comes Eudoxus, who found a way to define *proportion* (having the same ratio) for magnitudes instead of ratios of magnitudes.
- He invented the method of exhaustion which was used by Archimedes and Euclid to prove theorems that dealt with limits.
- Theodorus of Cyrene used Eudoxus approximation to prove irrationality of numbers in his spiral.

The Greeks' problems with infinitesimals/limits

The length of the stepped line is clearly $2s$ no matter how many steps there are. But as the number of steps increases, the stepped line seems to approach the diagonal, and the length of the diagonal is $\sqrt{2}s \neq 2s$.



The Dichotomy Paradox.

Anything in motion, must get halfway first to its destination. For example, to leave the room, you first have to get halfway to the door, then you have to get halfway from that point to the door, etc. No matter how close you are to the door, you have to go half the remaining distance before proceeding. Hence, there is no finite motion because the above process of always going half way while in motion is infinite.

> There is no motion, because what moves must arrive at the middle of its course before it reaches the end.

- Suppose length is 1 meter and object moves at 1 meter per second.
- It must reach halfway ($\frac{1}{2}$ meter from starting point) in $a_1 = \frac{1}{2}$ second. Let $t_1 = a_1$.
- From this halfway point, the object moves halfway to the end, which is $a_2 = \frac{1}{4}$ meters. The total time so far is $t_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4}$.
- We clearly have the following infinite sequences:

$$a_1, a_2, a_3, \ldots = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

$$t_1, t_2, \ldots = \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots \text{ where each } t_n = a_1 + a_2 + \cdots + a_n$$

- Zeno concluded that the total time which is the sum of an infinite sequence must be infinite and we can never reach our destination.
- We know that this is not the case, we can reach our destination in a finite time. So, where did Zeno get it wrong?

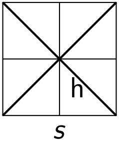
- First, note that
$$t_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$
- It follows that each t_n is less than 1, so the sequence of times never exceeds 1 second. This is why Zeno's conclusion is false.
- Moreover, $\frac{1}{2^n}$ gets smaller as n gets bigger. By taking a large enough value of n, we can make $\frac{1}{2^n}$ smaller than any small value we choose. Therefore:
$$\lim_{n \Rightarrow \infty} t_n = 1.$$
- Adding infinitely many numbers need not return an infinite. E.g.:
$$a_n = \frac{1}{2^n} \text{ and } \Sigma_{n=1}^{\infty} a_n = \Sigma_{n=1}^{\infty} \frac{1}{2^n}$$
- Hence,
$$2\Sigma_{n=1}^{\infty} a_n = 2a_1 + 2\Sigma_{n=2}^{\infty} \frac{1}{2^n} = 1 + \Sigma_{n=2}^{\infty} \frac{1}{2^{n-1}} = 1 + \Sigma_{n=1}^{\infty} \frac{1}{2^n} = 1 + \Sigma_{n=1}^{\infty} a_n.$$
- Therefore, $\Sigma_{n=1}^{\infty} a_n = 1$ and $lim_{n \to \infty} t_n = \Sigma_{n=1}^{\infty} a_n = 1$.

How did ancient Greeks deal with limits?

- Archimedes theorem giving the area of a circle
- A theorem of Euclid which says that the areas of circles are to each other as the squares of their radii.
- Both theorems were proved by a method that relied on Eudoxus theory of proportions which was a geometric theory designed to overcome the difficulties obtained from the discovery of the irrationals.
- For both Archimedes' theorem and Euclid's theorem, we need a general formula for the area of a regular polygon (i.e., a polygon where all angles are equal and all sides are equal).

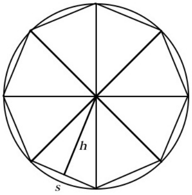
The area of a square of side s

Instead of taking area simply as s^2, take the bottom of the 4 triangles obtained by the diagonals. Altitude $h = \frac{1}{2}s$.



Area A of square $= 4\times$ area of triangle $= 4 \times \frac{1}{2}hs = \frac{1}{2}h(4s) = \frac{1}{2}hp$.
where p is the perimeter of the square.
Note also that $A = \frac{1}{2}hp = \frac{1}{2}\frac{s}{2}(4s) = s^2$.

The area of a regular octagon/circle



Now let us consider a regular octagon. If we divide it into triangles the same way, we get eight triangles, each of whose areas is $\frac{1}{2}hs$. If we take all eight triangles and note that here $p = 8s$, we get for the area $A = \frac{1}{2}h(8s) = \frac{1}{2}hp$.

The area of any regular polygon is one-half the altitude to a side times the perimeter, or $\frac{1}{2}hp$.

What about the circle?

- The above polygon was inscribed in the circle with circumference C.
- If we keep increasing the number of sides, the perimeter will approach the circumference C and the altitude will approach the radius r. This suggests that the formula for the area of a circle should be

$$A = \frac{1}{2} rC.$$

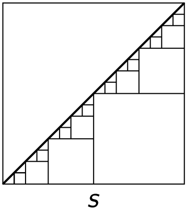
- And since π is defined to be the ratio of the circumference of a circle to twice its radius, we have

$$\pi = \frac{C}{2r},$$

- Hence

$$A = \frac{1}{2} r(2\pi r) = \pi r^2$$

- This must have seemed obvious to the ancient Greeks from an early period in the history of their geometry.
- But how could they prove it?
- At one time some of them argued that a circle is a regular polygon with infinitely many sides, but they eventually decided that this kind of reasoning is not immune to attacks by sophists.
- For just because regular polygons with an increasing number of sides seems to be approaching a circle, we are not automatically justified in deducing this formula for the area of a circle.
- They found evidence like this can be misleading.
- Recall the stepped line which wrongly gave the impression that $\sqrt{2}s = 2s$.



Euclid on Areas of Circles and Squares

- It took a long time for the proof that $A = \frac{1}{2}rC$ be given. Although this was obvious to the Greeks, a proof was hard to find.
- Before that proof was given, Euclid proved that the areas of circles have the same proportion (recall Eudoxus) as the squares on their diameters (Proposition 2 of Book XII of *Elements*).
- Its proof uses Proposition 1 of Book XII which states that *Similar polygons inscribed in circles are to one another as the squares on the diameters of the circles.*

PROPOSITION 1 OF BOOK XII OF THE *Elements*.
Similar polygons inscribed in circles are to one another as the squares on the diameters of the circles.

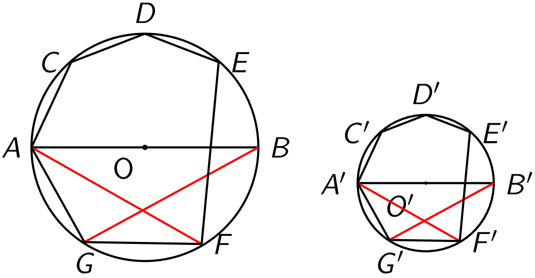
- Similar figures are those which have the same shape. In similar polygons the corresponding angles are equal and the corresponding sides all have the same proportion.

The areas of similar polygons are proportional to:

- The squares of their altitudes.
- The squares of their perimeters.
- The squares of any of their linear parts.

$\frac{p_1}{p_2} = \frac{h_1}{h_2}$ and $\frac{A_1}{A_2} = \frac{h_1 p_1}{h_2 p_2} = \frac{h_1}{h_2}\frac{p_1}{p_2} = \frac{h_1}{h_2}\frac{h_1}{h_2} = \frac{h_1^2}{h_2^2} = \frac{p_1^2}{p_2^2}$.

Proof of Proposition 1 of Book XII: Use above and the fact that AGB is similar to $A'G'B'$ below and hence $(\frac{AB}{A'B'})^2 = (\frac{AG}{A'G'})^2 = \frac{A_1}{A_2}$:



Proposition 2 of Book XII of the *Elements*.
Circles are to one another as the squares on the diameters.

- Let the circles have areas a and b respectively, and let the ratio of the squares of their diameters be k.
- Let the areas of the polygons inscribed in the circle with area a (resp. b) have areas $a_1, a_2, \dots$ (resp. $b_1, b_2, \dots$).
- We have $0 < a_1 < a_2 < \dots < a_n < \dots < a$ and $0 < b_1 < b_2 < \dots < b_n < \dots < b$.
- For each n, we have

$$k = \frac{a_n}{b_n}, \text{ so that } \frac{a_n}{k} = b_n.$$

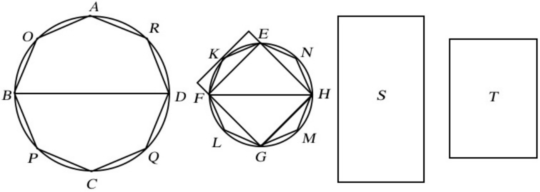
and

$$(a - a_{n+1}) < \frac{1}{2}(a - a_n), \qquad (b - b_{n+1}) < \frac{1}{2}(b - b_n).$$

- We want to prove

$$k = \frac{a}{b}.$$

Euclid's method is based on Eudoxus exhaustion



- If $k \neq \frac{a}{b}$, then $k = \frac{a}{S}$, where $S < b$ or $S > b$.
  - Suppose $S < b$. Choose N so that

$$b - b_N < b - S.$$

    The number N represents the number of times the number of sides of the inscribed polygon was doubled. Then

$$S < b_N.$$

    But

$$S = \frac{a}{k} > \frac{a_N}{k} = b_N,$$

    a contradiction.
  - Suppose $S > b$. This is similar to the above case with a and b reversed.

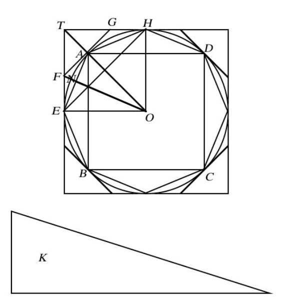
It follows that

$$k = \frac{a}{b}.$$

Proposition 1 of Archimedes's Book "Measurement of a Circle".

The area of any circle is equal to a right-angled triangle in which one of the sides about the right triangle is equal to the radius, and the other to the circumference of the circle.

Let $ABCD$ be the given circle, K the triangle described.



Then, if the circle is not equal to K, it must be either greater or less.

Archimedes' method is based on Eudoxus exhaustion

- Prove $A = \frac{1}{2}rC$ where r, A and C are radius, area and circumference.
- Let $K = \frac{1}{2}rC$ (the area of the triangle). If $A \neq K$, then:

  I. Suppose $A > K$.

  - Inscribe a square with side s_1, altitude to the side h_1, and perimeter p_1. The area of the square is $a_1 = \frac{1}{2}h_1 p_1$.
  - Now, double the number of sides of the inscribed polygon, and keep doubling it. For polygon n with side s_n, altitude to the side h_n, and perimeter p_n, the area is $a_n = \frac{1}{2}h_n p_n$.
  - From the geometry of the situation, we have that

    $$h_1 < h_2 < \ldots < h_n < \ldots r,$$

    $$p_1 < p_2 < \ldots < p_n < \ldots < C,$$

    and

    $$a_1 < a_2 < \ldots < a_n < \ldots < A.$$

  - Now choose N so that $A - a_N < A - \frac{1}{2}rC$.
  - It follows that $\frac{1}{2}rC < a_N$.
  - But since $h_N < r$, $p_N < C$, and $a_N = \frac{1}{2}h_N p_N$, we have $a_N < \frac{1}{2}rC$,
  - a contradiction.

II. Suppose, on the contrary, that $A < K$.

- Circumscribe a square with perimeter P_1; then the area is $A_1 = \frac{1}{2}rP_1$.
- Now double the number of sides of the circumscribed figure, and keep doing it. If, for the nth polygon, the perimeter is P_n, then the area is $A_n = \frac{1}{2}rP_n$.
- From the geometry, we have

$$C < \ldots < P_n < \ldots < P_2 < P_1$$

and

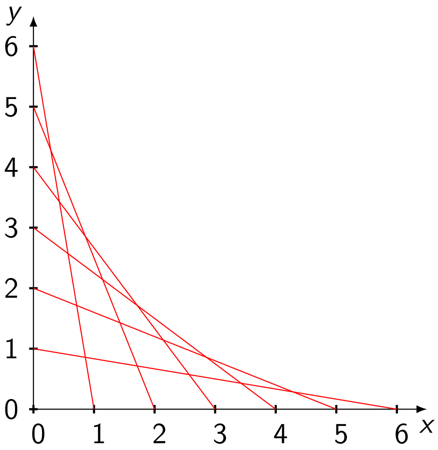
$$A < \ldots A_n < \ldots < A_2 < A_1.$$

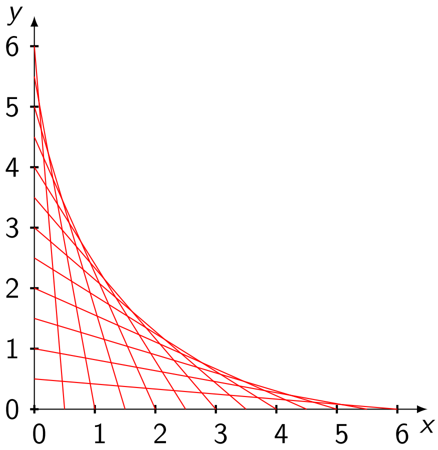
- Choose N so that $A_N - A < \frac{1}{2}rC - A$.
- Then $A_N < \frac{1}{2}rC$.
- But $C < P_N$ and $A_N = \frac{1}{2}rP_N$, so $\frac{1}{2}rC < A_N$,
- another contradiction.

- It follows that $A = K = \frac{1}{2}rC$.

- The ideas of Eudoxus can be used to develop a definition of the limit of a sequence and a function (without specifying that we are dealing with real numbers).
- Historically, the development of calculus and analysis in European mathematics developed before a definition of the real numbers.
- In fact, at the time of Descartes, Leibniz and Newton, it had not even been settled whether or not there were infinitely small quantities; This continued into the 19th century. For example, Cauchy **thought** that there were infinitely small quantities.
- Infinitesimals were *introduced* in 450 BC, *banned* by Eucledian mathematicians because of the problems they faced when reasoning about them, *banned again* in the 1630s by religious clerics in Rome.
- They still *flourished* in 17th century:
  - a curved line is made of infinitely small straight line segments, and
  - quantities that differ by an infinitely small quantity are equal.
- *crucial* for the development of calculus by Newton and Leibniz.
- *abandoned again* in 19th century due to their unclear logical status.
- *revived again* in 20th century.

A curved line is made of infinitely small straight line segments





The historic position of clerics against infinitesimals

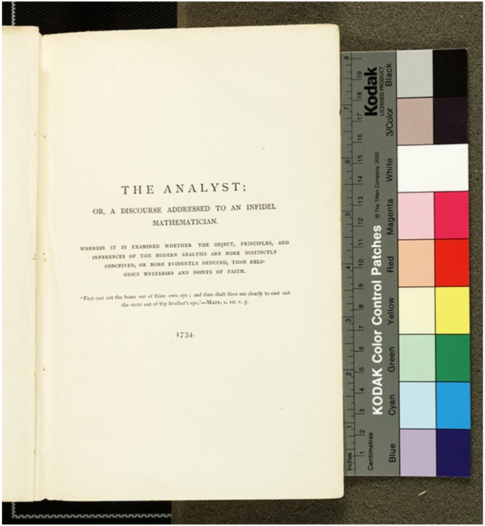
- To find the derivative $f'(2)$ at $x = 2$ of $y = f(x) = x^2$, assume $x \neq 2$.

  Then $$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(2)}{x - 2} = \frac{x^2 - 2^2}{x - 2} = \frac{(x+2)(x-2)}{x-2} = x + 2$$

- Since we are only able to conclude that the quotient is equal to x + 2 on the assumption that $x \neq 2$, we appear to have taken an illegal step.
- We justify this by saying that we are taking its limit as $x \to 2$:

$$\frac{dy}{dx} = \lim_{x \to 2} \frac{\Delta y}{\Delta x},$$

- Newton calls
$$\frac{dy}{dx} = \lim_{x \to 2} \frac{\Delta y}{\Delta x},$$
ultimate value or *value at instant of disappearance*.
- Bishop Berkeley called it *the ghost of a departed quantity*.



THE ANALYST;

OR, A DISCOURSE ADDRESSED TO AN INFIDEL MATHEMATICIAN.

WHEREIN IT IS EXAMINED WHETHER THE OBJECT, PRINCIPLES, AND INFERENCES OF THE MODERN ANALYSIS ARE MORE DISTINCTLY CONCEIVED, OR MORE EVIDENTLY DEDUCED, THAN RELIGIOUS MYSTERIES AND POINTS OF FAITH.

'First cast out the beam out of thine own eye; and then shalt thou see clearly to cast out the mote out of thy brother's eye.'—MATT. c. vii. v. 5.

1734.

- Foto from MAA.org.

Kepler's use of infinite versus Archimedes's use of exhaustion

Johannes Kepler used infinitesimals to calculate the area of an ellipse and viewed the circumference of a circle as an infinite sided regular polygon: for a circle of radius a and an ellipse of radiuses a and b:

- The ratio of each vertical line within the circle to the vertical line within the ellipse is a/b.
- The area of each of the circle/ellipse is the infinite sum of vertical lines contained in the circle/ellipse.

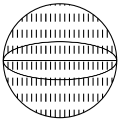


Figure 3: A circle of radius a and an ellipse of radiuses a and b. The area of each of the 2 shapes is the infinite sum of the dashed lines contained in it.

Infinitesimals and the birth of analysis

- Calculus formalizes the study of continuous change, while analysis provides it with a rigorous foundation in logic.
- The historic approach was to define limits, and develop the calculus without formally defining the real numbers.
- In the early nineteenth century, mathematicians began to question whether the deductive structure of the Elements was sufficient.
- And, they worried about the lack of rigorous foundations of the calculus (recall that the foundations of mathematics is a sound reasoning about the infinitesimal).
- Cauchy's ideas (19th cent.) of *function and limit* led to *rigorous* formulation of the calculus, limit/continuity/*real numbers*.
- Many controversies in analysis were solved by Cauchy. In particular with his precise definition of convergence in his *Cours d'Analyse*.
- 1872: Due to the more *exact definition of real numbers given by Dedekind*, the rules for reasoning with real numbers became even more precise.

- At school, after studying elementary algebra, you are introduced to geometry (*the study of shapes*) and trigonometry (*the study of side lengths and angles of triangles*) and then to more algebra.
- From arithmetic $\rightarrow$ elementary algebra $\rightarrow$ geometry& trigonometry, you move to a *pre-calculus course* which combines advanced algebra and geometry with trigonometry.
- After all this, you are introduced to *calculus*.
- Calculus (originally called *infinitesimal calculus*) is the mathematical study of continuous change.
- If Descartes had expressed rather than supressed infinitesimals and infinites in his method, he would have invented the calculus before Newton and Leibniz.

Descartes' innovative ruler-and-compass construction for *multiplying* lengths

- The ancient Greeks separated numbers, which are discrete, from continuous magnitudes. They did not use fractions to approximate continuous magnitudes, and they had different kinds of magnitudes for lengths, areas, volumes, angles, etc. They never multiplied two lengths to get another length.
- Descartes published a ruler-and-compass construction for multiplying two lengths to get a length.
- Descartes' ruler-and-compass construction for *multiplying* two lengths to get a length was innovative and it allowed Algebra to be a science concerned with numbers rather than geometric magnitudes.

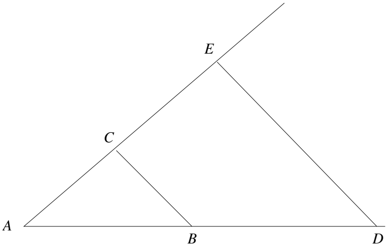


Figure 4: Descartes' Construction

The length of AB is a. On a line AC through A and at an angle to AB, let the length of AC be a unit, and construct E on the same line so that the length of AE is b. Join C and B with line segment BC, and construct a line through E parallel to BC; let this line intersect the extension of AB at D. Then triangles ABC and ADE are similar. Hence, AE is to AC as AD is to AB. I.e., $AD = ab$.

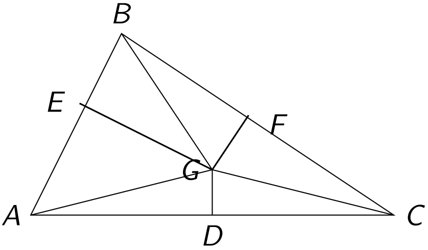
- Understanding analysis requires more mathematical sophistication than is required for understanding the calculus itself.

- arithmetic $\rightarrow$ elementary algebra $\rightarrow$ geometry and trigonometry $\rightarrow$ pre-calculus $\rightarrow$ calculus $\rightarrow$ analysis

- Textbooks give a number of rules for evaluating limits. There are different theories of limits and attempts continue at looking for new theories.
- Students find $\epsilon - \delta /\ \epsilon - N$ proofs of limits in analysis challenging.
- From our experience, an evolutionary and somewhat historic approach is helpful.

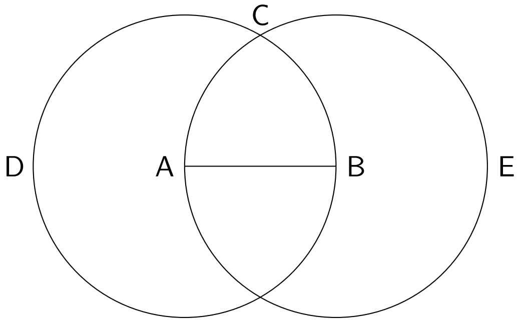
Along the way, they learn the need for rigor in mathematics

- Possible to Prove in Eucledian Geometry that Every Triangle is Isocele?



- A Problem in the Proof of Proposition 1 of Book I of Euclid's Elements.

> PROPOSITION 1. OF BOOK I OF THE *Elements*
>
> *To construct an equilateral triangle on a given finite straight line.*


C
D
A
B
E

What are the real numbers?

- A *field* is a set of objects $(S, +, \cdot)$ called quantities such that S is closed under $+$ and $\cdot$ and satisfies distributivity $a(b+c) = ab + ac$, and $+$ and $\cdot$ are commutative, associative, have identity elements (resp. 0 and 1) and inverses for each element (except for 0 under $\cdot$).
- A field is an *ordered field* if for all a, b, and c:
  - exactly one of $a < b$, $a = b$, and $b < a$ holds.
  - if $a < b$ and $b < c$, then $a < c$.
  - if $0 < a$ and $0 < b$, then $0 < a + b$ and $0 < ab$.
  - $a < b$ if and only if $0 < b + (-a)$.
- [AC.] AXIOM OF COMPLETENESS. Every nonempty set of quantities that has an upper bound has a least upper bound.
- **Real Numbers** $\mathbb{R}$ Our quantities form an ordered field that satisfies the Axiom of Completeness AC, and we will refer to them as *real numbers* and denote their collection by $\mathbb{R}$.

AL. ARCHIMEDES LAW. For any two quantities a and b where $b > a > 0$, there is a positive integer n such that $b < an$.

- An ordered field which also satisfies AL is called an *Archimedean ordered field*.
- None of $\mathbb{N}$ or $\mathbb{Z}$ is a field.
- $\mathbb{Q}$ is a field, is an orderd field and is an Archimedean ordered field.
- **Completeness implies the Archimedean Property** Assume a and b are real numbers such that $a > 0$. There is a positive integer n such that $an > b$.
- We can approximate real numbers by rational numbers.
  Density of rationals If a and b are any two real numbers with $a < b$, then there is a rational number r such that $a < r < b$.

Researchers in the 19th century continued to go deeper into numbers

- The formalisation of the real number and of mathematical analysis sparked further research into number systems and logic.
- 1895-1897: *Cantor began formalizing set theory* and made contributions to number theory.
- *Cantor's diagnolisation argument and the size of the natural numbers versus the size of the real numbers* will impact the size of what can be *computable* versus what cannot.
- Cantor proved that *algebraic numbers are countable*. and *the transcendental numbers are uncountable*.
- Later on (in the 1930s) Turing showed that the size of the computable functions is the size of the algebraic numbers, the smallest infinite.
- The size of the non-computable functions is the size of the transcendental numbers.
- This means that there are a lot more functions that are impossible to compute than there are computable functions.

The Language, Machine and Model of the Computable

- The work of Frege and Russell/Whitehead and Hilbert and Weyl and Ramsey led the way.
- In the late 1920s and early 1930, came Church (λ-calculus) and Curry (combinatory logic) and Turing (Turing machine) and gave us equivalent models of computation.
- λ-calculus and Turing machines were presented as a negative answer to Hilbert's Entscheidungsproblem.
- What motivated Curry was the problem he saw in Russell's substitution in Principia Mathematica.
- Curry noticed that the rule of substitution of well-formed formulas for **propositional variables** (not even bound variables) was considerably more tricky than the rule of detachment (which is equivalent to modus ponens).
- The complication Curry noticed in the rule of substitution of Principia is now considered to be the complication of its implementation by a computer program.