Transitive Closure and Inductive Reasoning

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Abstract

We argue that the concept of transitive closure is the key for understanding finitary inductive definitions and inductive reasoning. Hence the ability to define the transitive closure of any given relation and make appropriate inferences concerning it should be one of the most important features of every computerized system for doing Mathematics and every logical framework. We investigate languages with transitive closure operations and their expressive power. In particular, we show that with the simplest transitive closure operation one can define all recursive predicates and functions from 0, the successor function and addition, but addition is not definable from 0 and the successor function. We then show that in the presence of the operations of pairing and projection, the availability of the simplest transitive closure operation suffices for having all types of finitary inductive definitions (and corresponding inductive principles).

1 Introduction

It is well known that the natural numbers cannot fully be characterized using a first order language: This is an important mathematical fact about the relations between Logic and the natural numbers. However, the implications of this fact for our understanding of either of these two subjects are not so clear. Some take it as a demonstration that Logic alone is not sufficient for understanding our notion of a natural number (and so- as an absolute refutation of the logicist thesis). Others believe that it shows that there is *no* unique notion of a natural number: many interpretations of this notion exist, and all are equally acceptable. It seems to me that both interpretations are based on a false belief: that Logic and first-order logic are identical concepts. In my opinion the one thing that the mathematical fact mentioned above clearly shows is that this is *not* the case. The fact that first-order logic is not sufficient for characterizing the natural numbers which we all understand [1] simply means that first-order

[1]Even those who talk about *non*-standard natural numbers necessarily recognize the *standard* ones- and these are the entities I am talking about.

logic is incapable of telling the whole story that humane Logic has to tell.

The point of view just presented is, of course, not new. The reason that it is not very popular is another common belief: that in order to be able to give a categorical characterization of the natural numbers one has to use second-order logic. Indeed, the standard categorical axiomatization of the natural numbers, Peano's axioms, is implicitly based on this logic. Unfortunately, second-order logic is based on ontological commitments which are not universally accepted. It cannot therefore be taken as a part of the Logic which underlies all humane thoughts and reasoning. Rather, it is a part of Platonistic mathematics. Moreover: it is counterintuitive that in order to understand the notion of a natural number (which every child does) one needs to understand also the notion of an arbitrary set of natural numbers (which most people do not), but this precisely is what the use of second-order language for characterizing the natural numbers implies. It is no wonder that between the two horns of the dilemma: accepting second-order logic as a part of our universal Logic (and using it to characterize the natural numbers), versus accepting only first order logic as our Logic (and inferring that the natural numbers are not logically characterizable), most scholars choose the second alternative.

But is there really nothing between first-order and second-order logic which can justly be taken as a part of everybody's Logic? I believe that this is not the case. Here are two non-mathematical examples:

- I am a descendant of Jacob. Therefore my son is also a descendant of Jacob. Moreover: The descendants of my son will all be descendants of Jacob.
- If you make a sexual intercourse with someone who had made a sexual intercourse with someone who had made a sexual intercourse with ... someone who might had been a HIV carrier, then you might become a HIV carrier.[2]

Now everyone can (and does) understand the inferences made in the two examples given above. One does not need to be trained in mathematics or even to understand the natural numbers for that. This means that they are based on general *logical* principles. (I am convinced that most people will indeed classify the conclusions in the examples above as *logical* consequences of their assumptions!). These logical principles do not belong to first-order logic (as is well known). On the other hand the full power and ontological commitments

[2]Taken without permission from a T.V. campaign for safe sex.

of second-order logic is not needed for them. What is really involved in the two examples (and in many other examples of this sort) are the following two factors:

1. The construction of a certain new binary relation between objects from a given one. This relation is what mathematicians call "the transitive closure" of the old one.

2. The use of some general principles for inferring facts about objects which are related by the new relation,

The main thesis of this paper is that the concept of transitive closure is the real key for understanding inductive definitions and inductive reasoning from a *logical* point of view. Hence the ability to define the transitive closure of any given relation and make appropriate inferences concerning it is essential for any computerized system for doing Mathematics and for any logical framework. The results below provide some evidence for these claims.

2 Logics with a Transitive Closure Operation

2.1 The Basic Language

The language and semantics of the basic extension of first order logic with a transitive closure operation are defined as follows (see, e.g., [9, 8]):

Definition 1. *Let σ be a signature for a first-order language with equality. The language $L^1_{TC}(\sigma)$ is defined like the usual first-order language which is based on σ, but with the addition of the following clause:*

- *If φ is a formula, x, y are distinct variables, and t, s are terms, then $(TC_{x,y}\varphi)(t, s)$ is a formula. In this formula all the free occurrences of x and y in φ become bound.*

The semantics of the new type of formulas is defined as follows. Given a structure $\mathcal{D}$ for σ with domain D and an assignment v in D for the variables of the language, define:

- *$\mathcal{D}, v \models (TC_{x,y}\varphi)(t, s)$ iff there exist $a_0, a_1, \ldots, a_n \in D$ $(n \geq 1)$ such that $a_0 = v(t)$, $a_n = v(s)$, and $\mathcal{D}, v(x := a_i,\ y := a_{i+1}) \models \varphi$ for $i = 0, 1, \ldots, n-1$.*

Note 1. Given a formula φ, $(TC_{x,y}\varphi)$ is a new binary *predicate* symbol (in standard first-order languages only primitive predicate symbols are available!).

Note 2. In L^1_{TC} we really need only the propositional connectives and the TC operator, since the existential quantifier can be *defined* as follows:

$$\exists x\varphi =_{Df} \Big(TC_{u,v}\big(\varphi(u/x) \vee \varphi(v/x)\big)\Big)(t,s)$$

where t and s are terms. Any two terms would do, but in order for $\exists x\varphi$ to have less free variables than φ (in particular, in order for $\exists x\varphi$ to be a sentence in case x is the only free variable of φ), t and s should be (not necessarily distinct) closed terms. This is possible, of course, only if the signature σ has at least one constant.

Note 3. The reflexive transitive closure RTC can be defined as follows:

$$RTC_{xy}(t,s) \equiv_{Df} (t=s) \vee (TC_{xy}\varphi)(t,s) \ .$$

It is possible to take RTC as primitive, since the following is logically valid:

$$(TC_{xy}\varphi)(t,s) \Leftrightarrow \varphi(t/x, s/y) \vee \exists z\big(z \neq t \wedge (RTC_{x,y}\varphi)(t,z) \wedge (RTC_{x,y}\varphi)(z,s)\big) \ .$$

2.2 Three Examples

2.2.1 Archimedes' axiom

Archimedes' axiom in the standard signature of ordered fields says that given any $b > 0$ and a there is a natural number n such that $nb > a$. This axiom is not first-order, but using TC it can easily and naturally be formulated as follows:

$$\forall a \forall b\Big(b > 0 \rightarrow \exists z\big(z > a \wedge (TC_{x,y}x = y + b)(z,0)\big)\Big) \ .$$

Note that the natural numbers are not even mentioned in this formulation! (Of course, they are *definable* in this signature by: $N(n) \equiv_{Df} (TC_{x,y}x = y+1)(n,0)$.)

An interesting example of a famous theorem in Geometry, for which Archimedes' axiom (or the equivalent geometrical version) is needed, is Mascheroni-Mohr theorem according to which every point in the plane which can be constructed using a ruler and a compass (given a set of points) can be constructed using only a compass. An attempt of automatizing my new proof of this theorem in [1, 2] has been one of my major motivations for investigating $L^1_{TC}(\sigma)$ (see [3]).

2.2.2 The Natural Numbers

In $L^1_{TC}(\langle 0,S,+\rangle)$ one can define:

$$t < s \equiv_{Df} \big(TC_{x,y}\, y = S(x)\big)(t,s)\ .$$

The following finite set of axioms is then *categorical*, with $\mathbb{N}$ as the unique model:

N1.	$\forall x\big(S(x) \neq 0\big)$
N2.	$\forall x \forall y\big(S(x) = S(y) \Rightarrow x = y\big)$
N3.	$\forall x(x = 0 \vee 0 < x)$
N4.	$\forall x(x + 0 = x)$
N5.	$\forall x\big(x + S(y) = S(x+y)\big)$

Note that this system is of course *not* categorical if we define $t < s$ as $\exists z.S(z) + t = s$!

Theorem 1. *All the recursive functions are definable in* $L^1_{TC}(\langle 0,S,+\rangle)$.

Proof: Obviously, it is sufficient to show that Multiplication is definable in $L^1_{TC}(\langle 0,S,+\rangle)$. For this note that the division relation $(x \mid y \equiv_{Df} \exists z(x * z = y))$ is definable in $L^1_{TC}(\langle 0,S,+\rangle)$, since $\mathbb{N} \models x \mid y \Leftrightarrow (TC_{a,b}\, a + x = b)(0,y)$. Now, it is obvious that

$$\mathbb{N} \models x = y^2 \Leftrightarrow y \mid (x+y) \wedge S(y) \mid (x+y) \wedge \forall z(z < x+y \wedge y \mid z \Rightarrow \neg\big(S(y) \mid z)\big)\ .$$

$$\mathbb{N} \models x = y \cdot z \Leftrightarrow \exists u \exists v \exists w\Big(u = y^2 \wedge v = z^2 \wedge w = (y+z)^2 \wedge w = \big((u+v)+x\big)+x\Big)\ . \quad \square$$

Corollary 1. *The set of formulae of* $L^1_{TC}(\langle 0,S,+\rangle)$ *which are valid in* $\mathbb{N}$ *is not arithmetical.*

Corollary 2. *The set of logically valid formulae of* $L^1_{TC}(\langle 0,S,+\rangle)$ *is not arithmetical.*

Proof: A sentence φ of $L^1_{TC}(\langle 0,S,+\rangle)$ is valid in $\mathbb{N}$ iff $N1 \wedge \cdots \wedge N5 \to \varphi$ is logically valid.

Note 4. For a categorical characterization of the natural numbers $L^1_{TC}(\langle 0,S\rangle)$ and axioms N1-N3 already suffices. Unfortunately, the expressive power of this language is too weak (See Theorem 2 below). Note, however, that $L^1_{TC}(\langle 0,S\rangle)$ does suffice for our definition of $<$.

2.2.3 Mathematical Induction

Corollary 2 means that no complete and sound proof system for the logically valid formulae of L^1_{TC} is possible. Instead one should look for a computationally efficient sound proof system which will be strong enough for most mathematical needs. A particularly important valid *logical* rule of $L^1_{TC}(\sigma)$ which such a system should include is *induction*. Using a Gentzen-type format, a general form of this principle can be formulated as follows:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta, \psi(y/x)}{\Gamma, \psi(s/x), (TC_{x,y}\varphi)(s,t) \Rightarrow \Delta, \psi(t/x)}$$

where x and y should not occur free in Γ, Δ, and y should not occur free in ψ.

To see the connection between this rule and ordinary induction, assume that σ is a signature which contains 0 and S, and take φ to be $y = S(x)$. Substituting 0 for s we get:

$$\frac{\Gamma, \psi, y = S(x) \Rightarrow \Delta, \psi(y/x)}{\Gamma, \psi(0/x), \big(TC_{x,y} y = S(x)\big)(0,t) \Rightarrow \Delta, \psi(t/x)}$$

Using first order rules for $=$ and the definition of $<$ in $L^1_{TC}(\langle 0, S\rangle)$, this is equivalent to:

$$\frac{\Gamma, \psi \Rightarrow \Delta, \psi\big(S(x)/x\big)}{\Gamma, \psi(0/x), 0 < t \Rightarrow \Delta, \psi(t/x)}$$

Since obviously $\Gamma, \psi(0/x),\ 0 = t \Rightarrow \Delta, \psi(t/x)$ is a valid sequent, we get

$$\frac{\Gamma, \psi \Rightarrow \Delta, \psi\big(S(x)/x\big)}{\Gamma, \psi(0/x), 0 = t \vee 0 < t \Rightarrow \Delta, \psi(t/x)}$$

Using N3, this implies the validity of the following in the context of N1-N3:

$$\frac{\Gamma, \psi \Rightarrow \Delta, \psi\big(S(x)/x\big)}{\Gamma, \psi(0/x) \Rightarrow \Delta, \psi(t/x)}$$

This is exactly the form given to the induction rule by Gentzen in the classical [7] (in which the consistency of PA is proved).

3 Finitary Inductive Definitions

We have seen that induction as a method of proof is a part of the logic of $L^1_{TC}(\sigma)$. However, induction is used in Mathematics also as a tool for defining new concepts and relations (including functions, which we take here as a special type of relations). For this $L^1_{TC}(\sigma)$ is in general too weak:

Theorem 2. *$+$ is not definable in $L^1_{TC}(\langle 0, S\rangle)$.*

Proof: By a well-known result of Büchi ([5]. See also [4], p. 615]), $S1S$, the (monadic) second order theory of the successor function, is decidable. Now $L^1_{TC}(\langle 0, S\rangle)$ is interpretable in $S1S$, since a $(TC_{x,y}\varphi)(t,s)$ is equivalent to

$$\forall Z\Big\{\big[\forall u\big(\varphi(t/x, u/y) \to u \in Z\big) \wedge \forall x\forall y(x \in Z \wedge \varphi \to y \in Z)\big] \to s \in Z\Big\}$$

(where Z is a second order variable). This implies the decidability of the set of formulae of $L^1_{TC}(\langle 0, S\rangle)$ which are valid in $\mathbb{N}$. The theorem follows therefore from Corollary 1. $\Box$

One possible solution to the problem caused by Theorem 2 is to use stronger transitive closure operations. [9], e.g., considers the languages $L^n_{TC}(\sigma)$, in which $(TC^k_{x_1,\ldots,x_k,y_1,\ldots,y_k}\varphi)(\vec{t},\vec{s})$ is a formula whenever φ is a formula, $x_1,\ldots,x_k,y_1,\ldots,y_k$ $(k \le n)$ are $2k$ distinct variables, and $\vec{t},\vec{s}$ are k-vectors of terms. The semantics of $(TC^k_{\vec{x},\vec{y}}\varphi)(\vec{t},\vec{s})$ is defined as in the case $k = 1$, only this time we need to refer to vectors of length k of terms or variables. Now in $L^2_{TC}(\langle 0, S\rangle)$ addition (and so every recursive function) *is* definable, since

$$\mathbb{N} \models x = y + z \Leftrightarrow \big(RTC^2_{x_1,x_2,y_1,y_2} y_1 = S(x_1) \wedge y_2 = S(x_2)\big)(0, y, z, x)$$

However, the use of TC^k is not so natural. Moreover: its implementation is a serious problem for proof checkers which are based on strict discipline of types, like de Bruijn's Automath or LF-style Logical Frameworks (see [10]). Such frameworks allow only finite signatures, and so it is impossible to directly include in them all the TC^ks.[3] An obvious better solution is to allow the construction of *pairs* as legitimate terms of the language. The idea is that (t, s) should be a term whenever t and s are terms. Using pairs all the TC^k can be reduced to TC^1 in a trivial way.

We now show our principal result: pairs, the two projection functions, and the basic TC (i.e. TC^1) together suffice for having *all* types of finitary inductive definitions at our disposal! In doing so we shall rely on Feferman's analysis and characterization of finitary inductive definitions in [6].

Let σ include (at least) one constant 0, a binary function symbol $Pair$, and the unary function symbols P_1 and P_2 (we shall usually write (x, y) instead of $Pair(x, y)$). Let $\mathcal{D}$ (with domain D) be a structure for σ in which $P_1(Pair(x, y)) = x$ and $P_2(Pair(x, y)) = y$ for all x and y. Denote by $\mathcal{K}$ the set of subsets of D which are definable by some formula of $L^1_{TC}(\sigma)$. As noted by Feferman in [6], every subset A of D induces an n-ary relation $A^{(n)}$ on D, defined by: $A^{(n)}(x_1,\ldots,x_n) \Leftrightarrow (x_1,\ldots,x_n) \in A$ (where $(x_1,\ldots,x_n)$ is an abbreviation of $(\cdots((x_1, x_2), x_3)\cdots))$. Conversely: every n-ary relation $A^{(n)}$ on D is induced by some $A \subseteq D$. For $B \subseteq D$ let $TC(B)$ be the subset of D which induces the transitive closure of the binary relation induced by B. Obviously, $TC(B) \in \mathcal{K}$ for all $B \in \mathcal{K}$.

[3]The real source of the problem is that the generalized transitive closure operation does not have a unique arity. Hence any system in which every constant should have a fixed arity can directly handle it only by officially using an infinite number of primitive symbols.

Theorem 3. *$\mathcal{K}$ is closed under finitary inductive definitions: If $A_0, \dots, A_m, B_0, \dots, B_p$ are all in $\mathcal{K}$, then so are the least $X_0, \dots, X_m$ which satisfy the following conditions:*

(1) $A_i \subseteq X_i \quad (0 \leq i \leq m)$.

(2) If $a_1 \in X^j_{k_1}, \dots, a_{n_j} \in X^j_{k_{n_j}}$ *and* $(b, a_1, \dots, a_{n_j}) \in B_j$ *then* $b \in X_j$ $(0 \leq j \leq p)$.

Proof: Using Feferman's constructions and arguments in [6] (concerning his system FS_0), it is not difficult to prove the theorem once we show that $\mathcal{K}$ is closed under Feferman's operation I_2, where for $A, B \subseteq D$, $I_2(A, B)$ is the least $X \subseteq D$ such that:

(i) $A \subseteq X$

(ii) If $x_1 \in X$, $x_2 \in X$, and $((y, x_1), x_2) \in B$, then $y \in X$.

To show the closure of $\mathcal{K}$ under I_2, let $list(X)$ be $\{0\} \cup \{(0, x_1, \dots, x_n) \mid x_1, \dots, x_n \in X\}$. The main idea of the proof is to show that given A and B, $\mathcal{K}$ contains the set of lists which can be viewed as "proofs" that their last element is in $I_2(A, B)$. Obtaining then $I_2(A, B)$ itself is easy, since for any element a of $list(X)$ both n and $x_1, \dots, x_n$ are uniquely determined, and $x_n = P_2 a$.

The details are given in the following definitions and easily shown facts:

- $I_1(A, B) = \{x \mid x \in A \vee \exists y \in A.(x, y) \in TC(B)\}$. Obviously, $\mathcal{K}$ is closed under I_1.

- $list(A) = I_1(\{0\}, \{(x, y) \mid P_1 x = y \wedge P_2 x \in A\})$. Hence $list(A) \in \mathcal{K}$ whenever $A \in \mathcal{K}$.

- Let $1^- Perm$ be the set of all pairs of the form:

$$((x_1, \dots, x_k, a, b, y_1, \dots, y_n),\ (x_1, \dots, x_k, b, a, y_1, \dots, y_n))$$

where $k > 0$ and $n \geq 0$. Then $1^- Perm \in \mathcal{K}$. Indeed, $1^- Perm = I_1$ (*Switch, Enlarge*), where

$$Switch = \{(x, y) \mid P_1 P_1 x = P_1 P_1 y \wedge P_2 P_1 x = P_2 y \wedge P_2 x = P_2 P_1 y\}$$
$$Enlarge = \{(x, y) \mid P_1 P_1 x = P_1 y \wedge P_1 P_2 x = P_2 y \wedge P_2 P_1 x = P_2 P_2 y\} \ .$$

Here $Switch$ is the set of all pairs of the form $((x_1, \dots, x_k, a, b), (x_1, \dots, x_k, b, a))$ where $k > 0$, while $Enlarge$ corresponds to the rule: $\frac{(a,b) \in 1^- Perm}{((a,v),(b,v)) \in 1^- Perm}$.

- Let $Perm$ be the set of all pairs of the form: $((a, x_1, \ldots, x_n), (a, x_{\pi(1)}, \ldots, x_{\pi(n)}))$ where π is a permutation of $\{1, \ldots, n\}$. Then $Perm \in \mathcal{K}$, since $Perm = I_1(id^*, 1^- Perm)$, where $id^* = \{(x, x) \mid x = (P_1 x, P_2 x)\}$.

- Let $Proof(A, B)$ be the set of lists $(0, x_1, \ldots, x_n)$ for which there exists a permutation π of $\{1, \ldots, n\}$ such that for every $1 \leq i \leq n$, either $x_{\pi(i)} \in A$ or there exists j, k such that $\pi(j), \pi(k) < \pi(i)$ and $(x_{\pi(i)}, x_{\pi(j)}, x_{\pi(k)}) \in B$. Then $Proof(A, B) \in \mathcal{K}$, since $Proof(A, B) = I_1(list(A), B^*)$, where

$$B^* = Perm \cup \{(x, y) \mid P_1 x = y \wedge (P_2 x, P_2 y, P_2 P_1 y) \in B\} .$$

  Here in the second disjunct of B^* x is obtained from y by adding to y the result of applying the rule B to the last two components of y.

- $I_2(A, B) = \{y \mid \exists x \in Proof(A, B). y = P_2 x\}$. Hence $I_2(A, B) \in \mathcal{K}$ whenever $A, B \in \mathcal{K}$.

Note 5. Feferman has taken I_2 as a primitive of his FS_0. A proof similar to that of Theorem 3 can show that he could have taken instead just I_1 or TC (applied to classes).

4 Conclusion and Further Research

A great deal of attention has already been given in the past to logics with transitive closure operations in the context of Finite Model Theory (FMT). Our own interest, however, is in Logics which are meaningful and applicable to *every* structure. In fact, infinite structures are more interesting and important from the point of view of formalizing Mathematics. Moreover: most of the research in FMT has indeed been concentrated on model theory (with an emphasis on fixed-point logics). However, for formalizing Mathematics we need useful *proof* systems. I believe that from this point of view logics which are based on the use of transitive closure will prove superior to fixed-point logics, although the expressive power of the latter is stronger (for finite structures). As noted above, our next major goal is indeed to develop a computationally efficient sound proof system for L^1_{TC} which will be strong enough for most mathematical needs. We further believe, in fact, that the corresponding logic will suffice for most of applicable Mathematics, and no extra existential assumptions are needed. Showing this belief to be true is a further future project.

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