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Quantisation of monopoles with non-abelian magnetic charge

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Abstract

Magnetic monopoles in Yang-Mills-Higgs theory with a non-abelian unbroken gauge group are classified by holomorphic charges in addition to the topological charges familiar from the abelian case. As a result the moduli spaces of monopoles of given topological charge are stratified according to the holomorphic charges. Here the physical consequences of the stratification are explored in the case where the gauge group $SU(3)$ is broken to $U(2)$. The description due to A. Dancer of the moduli space of charge two monopoles is reviewed and interpreted physically in terms of non-abelian magnetic dipole moments. Semi-classical quantisation leads to dyonic states which are labelled by a magnetic charge and a representation of the subgroup of $U(2)$ which leaves the magnetic charge invariant (centraliser subgroup). A key result of this paper is that these states fall into representations of the semi-direct product $U(2) \ltimes \mathbf{R}^4$. The combination rules (Clebsch-Gordan coefficients) of dyonic states can thus be deduced. Electric-magnetic duality properties of the theory are discussed in the light of our results, and supersymmetric dyonic BPS states which fill the $SL(2, \mathbf{Z})$-orbit of the basic massive W-bosons are found.

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1. Introduction

The study of magnetic monopole solutions in spontaneously broken gauge theories, sparked off more than twenty years ago by 't Hooft's and Polyakov's discovery of the eponymous monopole solution in SU(2) Yang-Mills-Higgs theory [1], has progressed from the classification and in some cases explicit construction of monopoles via the description of the spaces of solutions (the moduli spaces) to, more recently, the discussion of classical and quantised dynamics of monopoles. Perhaps not surprisingly, most progress has been made in the theory originally considered by 't Hooft and Polyakov where, in a special limit called the BPS limit, the understanding of the geometry of the classical moduli spaces could be used rigorously to establish the existence of infinitely many (supersymmetric) quantum bound states of magnetic monopoles [2][3]. These bound states, which are typically dyonic, are related to the electrically charged W-bosons of the theory via electric-magnetic duality or S-duality and their existence confirms the possibility of formulating Yang-Mills theory in infinitely many equivalent ways, all related by S-duality and each having different particles as fundamental degrees of freedom.

As far as the spectrum of magnetically and electrically charged particles is concerned the step from Yang-Mills-Higgs theory with gauge group $SU(2)$ broken to $U(1)$ to a general gauge group G broken to some subgroup H is not of great qualitative difficulty provided the group H is abelian. Indeed, in the case $G = SU(N)$ and $H = U(1)^{N-1}$ (a maximal torus of G) much is known about monopole solutions, their moduli spaces, their quantum bound states and the action of electric-magnetic duality, even quantitatively [4]. When H is non-abelian, however, it was already pointed out more than ten years ago by a number of authors that qualitatively new problems arise when attempting physically to interpret parameters of multimonopole solutions and when discussing dyonic excitations of monopoles. The many interesting physical problems unearthed by these discussions, however, were largely ignored in the more mathematical treatment of monopoles and their moduli spaces in recent years. In attempting the final of the steps outlined above, that of trying to apply the mathematical understanding of monopole moduli spaces to the study of classical and quantum dynamics of monopoles, these problems naturally return. In particular they beset any attempt of understanding electric-magnetic duality in these theories. Let us therefore briefly review the issues involved.

The allowed values of magnetic charges in non-abelian gauge theories are restricted by the generalised Dirac quantisation condition see [5], [6], and also [7]. A naive application of this condition leads to the representation of the allowed magnetic charges as points on the dual of the weight lattice of the full gauge group G. Thus, if we denote the rank of G by R, magnetic charges correspond to vectors in a R-dimensional lattice, usually called magnetic weight vectors. This general condition does not reflect the specifics of the symmetry breaking. On the other hand there is a topological classification of magnetic charges which depends crucially on the breaking of G down to the exact residual gauge group H. The topological charges are elements of the second homotopy group $\Pi_2(G/H)$ ($= \Pi_1(H)$ if G is simply connected) and thus depend strongly on the connectedness of H. If one breaks the symmetry to a maximal torus $H \cong U(1)^R$ of G (maximal symmetry breaking) the classification resulting from the Dirac condition agrees with the topological classification: all R components of the magnetic weight vectors are topologically conserved. Now consider the degenerate situation where one does not break maximally and the exact group H is non-abelian. The magnetic weight vectors now have more components than the number of topologically conserved charges, raising the question of which - if any - relevance the remaining components (called the non-abelian components) might have. The answer is rather subtle and depends on additional assumptions. If one carries the Brandt-Neri reasoning [8] over to the unbroken group H one might expect that configurations whose magnetic weight vectors have non-abelian components will decay to configurations whose non-abelian components are, in a suitable sense, minimal. There is no topological obstruction to doing so. However, if one considers the theory in the BPS limit, shedding non-abelian charge in this way does not lower the energy. Mathematical analysis has revealed that there are neutrally stable solutions which have magnetic weights with non-minimal non-abelian components (restricted to lie in a certain range). In this situation the magnetic weights regain some of their glory. Mathematically they characterise holomorphic properties of the solution and they (or more precisely certain equivalence classes of them) are called holomorphic charges in the mathematical literature. Thus one may say that in general the magnetic weight vectors have topological and holomorphic components.

To illustrate the general discussion consider the simplest non-trivial example where $G = SU(3)$ and $H = U(1) \times U(1)$ or $H = U(2)$ for, respectively, maximal or minimal symmetry breaking. In Fig. 1 we show the magnetic weight lattice, spanned by the two simple roots $\vec{\beta}_1$ and $\vec{\beta}_2$, together with the direction $\vec{h}$ of the Higgs field which determines the symmetry breaking pattern. For each point in the weight lattice we indicate the dimensionality of the corresponding space of monopole solutions, called moduli space and to be defined more precisely in the main part of the paper. In the case of maximal symmetry breaking there are non-trivial moduli spaces for all positive magnetic weights (there are also moduli spaces for negative magnetic weights, containing anti-monopoles, but we do not consider these here). In the case of minimal symmetry breaking non-trivial moduli spaces only occur inside the cone shown in Fig. 1.b, where the topological component of the magnetic weight vectors is plotted vertically and the holomorphic component horizontally. The geometrical structure of these spaces is intricate. Each topological sector corresponds to a connected moduli space which consists of different strata of varying dimensions, with each stratum labelled by an (integer) holomorphic charge. There are one-parameter families of solutions where the holomorphic charge jumps from one discrete value to the next as the parameter varies continuously. These mathematical facts have so far not been interpreted physically. What is the physical meaning of the parameters which appear and disappear along the journey through the moduli space?

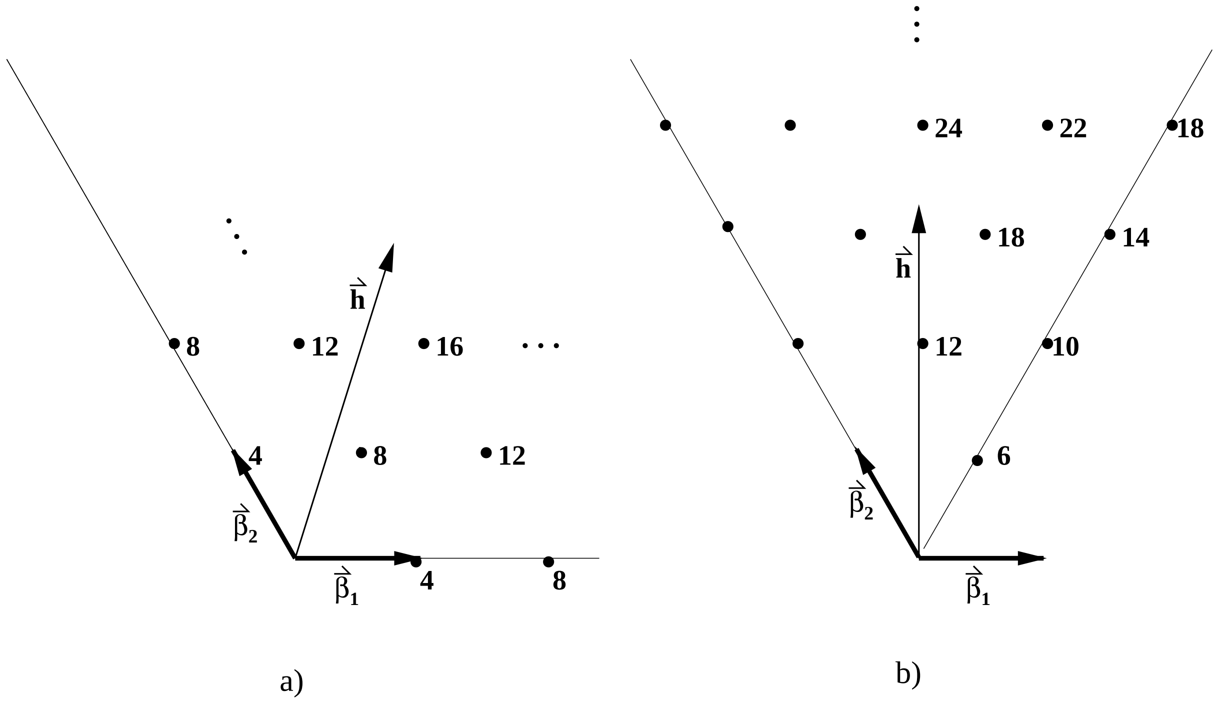

Fig. 1

a) *Moduli spaces and their dimensions for $SU(3)$ monopoles with maximal symmetry breaking*
b) *Strata of moduli spaces and their dimensions for $SU(3)$ monopoles with minimal symmetry breaking*

There is a related question which has attracted attention for a long time but which has never been resolved satisfactorily. This is the question of how the exact symmetry group H is realised in the various magnetic sectors. If the magnetic weight vector has non-abelian components it is not invariant under the action of H but instead sweeps out an orbit, which we call the magnetic orbit in this paper. All magnetic weight vectors on such a magnetic orbit (which includes the Weyl orbit of the magnetic weight vector) are allowed by the Dirac quantisation condition. In a given theory, however, different sorts of orbits arise. In the minimally broken $SU(3)$ theory, the magnetic weight vectors on the vertical axis are invariant under the $U(2)$ action, but all other magnetic weight vectors sweep out orbits which are two-spheres of quantised radii in the Lie-algebra of $SU(3)$. It is well known that "electric" excitations of a monopole with given magnetic weight do not, as one might naively expect, fall into representations of the exact group H but only form representations of the centraliser subgroup of the magnetic weight vector. This implies that the semi-classical dyon spectrum in a gauge theory with a non-abelian unbroken gauge group H displays an intricate interplay between magnetic and electric quantum numbers. In the $SU(3)$ example we see that electric excitations of monopoles with magnetic weight vectors on the vertical axis form $U(2)$ representations while for the other magnetic weights the electric excitations only carry representations of $U(1) \times U(1)$. Moreover, in the latter case one of the $U(1)$ groups depends on the point of the magnetic orbit to which the magnetic weight vector belongs. In other words monopoles in this sector may be charged with respect to different $U(1)$ subgroups. Clearly these matters have to be reconciled at the quantum level, where one expects a single algebraic framework which allows for a unified characterisation of both magnetic and electric properties of the states, and which furthermore allows one to combine the various sectors in some sort of tensor product calculus. Finally, such a framework can be expected to be a crucial ingredient in a discussion of electric-magnetic duality properties in gauge theories with non-abelian unbroken gauge group.

Having reviewed the physical problems we address in this paper, we can now outline our strategy for tackling them by giving the plan of this paper. We confine attention to monopoles in $SU(3)$ Yang-Mills-Higgs theory. The generalisation of arguments to general gauge groups and symmetry breaking patterns will be presented in a separate paper [9].

We begin in Sect. 2 with a review of classical monopole solutions in $SU(3)$ Yang-Mills-Higgs theory. Then we go on, in Sect. 3, to define classical configuration spaces and the moduli spaces of monopole solutions of the Bogomol'nyi equation. We review the stratification of the moduli spaces and discuss some of their geometrical properties. Monopole solutions in the so-called smallest stratum can be obtained by embedding $SU(2)$ monopole solutions in the $SU(3)$ theory, and this is described in Sect. 4. We then take a break from the mathematics of moduli spaces and in two short sections we formulate the two physical problems reviewed above - that of implementing the exact $U(2)$ symmetry and that of physically understanding the monopole moduli - in sharper mathematical language. For the rest of the paper we focus on monopoles of topological charge one and two. We review the rational map description of monopoles in Sect. 7 and in Sect. 8 we describe the work of Andrew Dancer who investigated the 12-dimensional large stratum of the charge two monopole moduli space in great detail. The physical interpretation of Dancer's moduli space is tricky because the mathematically most convenient description of the space is in terms of objects such as rational maps or Nahm data which are related to the actual monopole fields via mathematically complicated transformations. Using a combination of the different descriptions we are able to give a physical interpretation of the moduli of charge two monopoles in Sect. 9. It had been noted long ago that the interpretation of the parameters in terms of the zero-modes of two individual monopoles is problematic and we will see that individual monopoles indeed have new zero-modes when combined into a multi-monopole configuration. We argue that these new zero-modes are related to the possibility of the individual monopoles having non-vanishing dipole moments when part of a multimonopole configuration.

Sect. 10 is the key section of our paper. Here we present a detailed study of semi-classical dyonic excitations of monopoles. Dyonic quantum states are realised as wavefunctions on the strata of the moduli space and have to satisfy certain superselection rules. The dependence of the $U(2)$ action on the moduli space on the strata feeds trough to the dyonic wavefunctions in just the expected way: In the large stratum the $U(2)$ action is free and differentiable and as a result the monopoles carry full $U(2)$ representations. In all the other strata, however, the non-abelian magnetic charge obstructs the $U(2)$ action and the monopoles only carry representations of the $U(1) \times U(1)$ centraliser subgroup of the non-abelian magnetic charge. Remarkably we find that general dyonic states may be interpreted as elements of representations of the semi-direct product of $U(2) \ltimes \mathbf{R}^4$. Although this group does not act on the moduli spaces, it does have a natural action on wavefunctions on the moduli spaces. Thus quantum states, realised as wavefunctions on the moduli spaces, may be organised into representations of $U(2) \ltimes \mathbf{R}^4$. Such representations are labelled by a magnetic label specifying the orbit of the magnetic charge under the $U(2)$ action and by an electric label specifying the representation of the magnetic charge's centraliser subgroup. In fact, the interplay between orbits and centraliser representations is familiar in the theory of induced representations of regular semi-direct products (most famously in the representation theory of the Poincaré group). Not all representations of $U(2) \ltimes \mathbf{R}^4$ arise, however. Rather, the Dirac quantisation condition selects certain representations and also imposes restrictions on the representations which may be multiplied with each other. With these restrictions we get a complete and consistent description of the dyon spectrum, and can compute the Clebsch-Gordan coefficients for tensor products of dyonic states. We emphasise that at this point the full magnetic orbits play a crucial role. A number of authors have argued that only the orbits of the magnetic weight vectors under the Weyl group of the exact symmetry group H are physically relevant. Here we shall see that one cannot understand the fusion of two charge one monopoles carrying $U(1)$ charges into a charge two monopole carrying $U(2)$ charge without including the magnetic orbit in the discussion.

In Sect. 11 we turn to a discussion of electric-magnetic duality or, more generally, S-duality in $SU(3)$ Yang-Mills-Higgs theory. By S-duality we mean the generalisation of original electric-magnetic duality conjecture by Montonen and Olive [10] which takes into account the θ-term and the resulting Witten effect and which is believed to be an exact duality of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. This version has so far almost exclusively been studied in the case where the unbroken gauge group is abelian (see, however, [11] and our comments at the end of Sect. 11). If the unbroken gauge group H is non-abelian, on the other hand, a generalisation of the abelian electric-magnetic duality conjecture was formulated by Goddard, Nuyts and Olive in [6]. According to the GNO conjecture the magnetically charged states of the theory fall into representations of the dual group $\tilde{H}$ of the unbroken gauge group H. The full symmetry group of the theory would then be $H \times \tilde{H}$. However, this conjecture is not compatible with the results of our investigation of the semi-classical dyon spectrum. In treating magnetic and electric properties as independent the representation theory of the GNO group $H \times \tilde{H}$ does not correctly account for the interplay between magnetic and electric charges. Our semi-direct product group $H \ltimes \mathbf{R}^D$ ($D = \dim(H)$), by contrast, accurately captures this interplay and moreover leads to Clebsch-Gordan coefficients for the combination of dyonic states which are consistent with the realisation of the dyonic states as wavefunctions on monopole moduli spaces.

We therefore study the possibility of defining an action of S-duality on the representations of $H \ltimes \mathbf{R}^D$. In a natural implementation of this idea the electric-magnetic duality operation exchanges the two sorts of labels which characterise the representations of $H \ltimes \mathbf{R}^D$, namely the magnetic orbits labels and the labels of electric centraliser representations. To test this possibility we set up a $\mathcal{N}=4$ supersymmetric quantisation scheme. We show that the dyonic BPS states which are the S-duality partners of the massive W-bosons in the minimally broken $SU(3)$ theory can be found by a suitable embedding of BPS states in $SU(2)$ Yang-Mills-Higgs theory. Nonetheless a number of open problems remain, and these are discussed in the final Sect. 12.

2. A review of $SU(3)$ monopoles

A monopole solution of $SU(3)$ Yang-Mills-Higgs theory with coupling constant e in the Bogomol'nyi limit is a pair (A_i, Φ) of a SU(3) connection A_i and an adjoint Higgs field Φ on $\mathbf{R}^3$ satisfying the Bogomol'nyi equations

$$D_i \Phi = B_i \tag{2.1}$$

as well as certain boundary conditions, to be specified below. We use the notation $\mathbf{r}$ for a vector in $\mathbf{R}^3$, ∂_i for partial derivatives with respect to its Cartesian components, which we also sometimes write as (x, y, z), and $r = |\mathbf{r}|$ for its length. In writing (2.1) we have also used the usual notation $D_i = \partial_i + eA_i$ for the covariant derivative and B_i for the non-abelian magnetic field:

$$B_i = \frac{1}{2}\epsilon_{ijk}\left(\partial_j A_k - \partial_k A_j + e[A_j, A_k]\right). \tag{2.2}$$

To discuss solutions of this equation, in particular the precise boundary conditions, we need to introduce some notation for the Lie algebra $su(3)$ of $SU(3)$.

A Cartan subalgebra (CSA) of $su(3)$ is given by the set of diagonal, traceless 3×3 matrices, and for definiteness we choose the following basis $\{H_1, H_2\}$, normalised so that $\mathrm{tr}\,(H_\mu H_\nu) = \frac{1}{2}\delta_{\mu\nu}$, $\mu, \nu = 1, 2$.

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{2.3}$$

Often we also write $\vec{H} = (H_1, H_2)$. We complement the Cartan generators by ladder operators $E_{\vec{\beta}}$, one for each of the roots $\vec{\beta} = (\beta_1, \beta_2)$, thus obtaining a Cartan-Weyl basis of $su(3)$:

$$[H_\mu, E_{\vec{\beta}}] = \beta_\mu E_{\vec{\beta}} \quad \text{and} \quad [E_{\vec{\beta}}, E_{-\vec{\beta}}] = 2\vec{\beta}\cdot\vec{H}. \tag{2.4}$$

As simple roots one may take for example

$$\vec{\beta}_1 = (1, 0) \quad \text{and} \quad \vec{\beta}_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}). \tag{2.5}$$

All roots can be written as integer linear combinations of these with either only positive or only negative coefficients. It is also important for us that for any given root $\vec{\beta}$ the elements $\vec{\beta}\cdot\vec{H}, E_{\vec{\beta}}, E_{-\vec{\beta}}$ satisfy the commutation relations of $SU(2)$ with $E_{\pm\vec{\beta}}$ playing the roles of raising and lowering operators. In particular, we will work with the $SU(2)$ algebras associated to the simple roots, so we write down the associated generators explicitly. For $\vec{\beta}_1$ we define

$$I_3 = \vec{\beta}_1\cdot\vec{H} = H_1, \quad I_+ = E_{\vec{\beta}_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_- = E_{-\vec{\beta}_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.6}$$

and introduce $I_1 = (I_+ + I_-)/2$ and $I_2 = (I_+ - I_-)/2i$, as well as the vector notation $\mathbf{I} = (I_1, I_2, I_3)$. For $\vec{\beta}_2$ we write similarly

$$U_3 = \vec{\beta}_2\vec{H} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad U_+ = E_{\vec{\beta}_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_- = E_{-\vec{\beta}_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.7}$$

and also introduce $U_1 = (U_+ + U_-)/2$ and $U_2 = (U_+ - U_-)/2i$. Finally we define the "hypercharge" operator

$$Y = \frac{2}{\sqrt{3}} H_2 \tag{2.8}$$

and note that Y, I_1, I_2, I_3 satisfy the $u(2)$ Lie algebra commutation relations. Later we will need an explicit parametrisation of the $U(2)$ subgroup of $SU(3)$ generated by I_1, I_2, I_3 and Y, so we define Euler angles by writing an arbitrary element P in that subgroup as

$$P(\chi, \alpha, \beta, \gamma) = e^{i\chi Y} e^{-i\alpha I_3} e^{-i\beta I_2} e^{-i\gamma I_3}. \tag{2.9}$$

The ranges for the angles are $\chi \in [0, 2\pi), \alpha \in [0, 2\pi), \beta \in [0, \pi)$ and $\gamma \in [0, 4\pi)$, supplemented by the $\mathbf{Z}_2$-identification $(\chi, \gamma) \sim (\chi + \pi, \gamma + 2\pi)$.

The boundary condition we impose on solutions of (2.1) are of different types. The first sets the symmetry breaking scale:

$$|\phi|^2 \to \frac{1}{2}v^2 \quad \text{for } r \to \infty, \tag{2.10}$$

where we have used the $su(3)$ norm $|\phi|^2 = -\frac{1}{2}\mathrm{tr}\phi^2$ The second stems from the wish to keep the potential energy

$$E(A_i, \Phi) = -\frac{1}{2}\int d^3x \, \mathrm{tr}\left(D_i\Phi^2\right) + \mathrm{tr}\left(B_i^2\right) \tag{2.11}$$

finite. To this end we impose

$$|B_i| = |D_i\Phi| = \mathcal{O}(\frac{1}{r^2}) \qquad \text{for large } r. \tag{2.12}$$

Finally we demand that the Higgs field has the following form along the positive z-axis:

$$\Phi(0,0,z) = \Phi_0 - \frac{G_0}{4\pi z} + \mathcal{O}(\frac{1}{z^2}), \tag{2.13}$$

where Φ_0 is a constant element of $su(3)$, chosen to lie in the CSA so that we may define $\vec{h}$ via $\Phi_0 = i\vec{h}\cdot\vec{H}$. It follows from the Bogomol'nyi equation (2.1) that G_0 commutes with Φ_0 [12], and this makes it possible to also demand that G_0 is in the CSA. However, we will not impose this requirement at this stage, which in the mathematical literature is called framing.

The constant part Φ_0 determines the symmetry breaking pattern: the unbroken gauge group is the subgroup of $SU(3)$ which commutes with Φ_0 (its centraliser). If $\vec{h}$ has a non-vanishing inner product with both the simple roots (2.5) the symmetry is broken maximally to the $U(1) \times U(1)$ group generated by the CSA. In that case Φ_0 has three distinct eigenvalues. If $\vec{h}$ is orthogonal to either of the simple roots, say $\vec{\beta}_1$, then the symmetry is broken to the $U(2)$ subgroup of $SU(3)$ defined earlier. We say the symmetry is minimally broken. In this case, which is our main concern in this paper, Φ_0 has a repeated eigenvalue, and for definiteness we shall then assume the following form

$$\Phi_0 = ivH_2, \tag{2.14}$$

where the vacuum expectation value v sets the scale for the masses of all particles in the theory.

The Bogomol'nyi equation relates the coefficient G_0 to the coefficient of the long range part of the magnetic field so that along the positive z-axis

$$B_3(0,0,z) = \frac{G_0}{4\pi z^2} + \mathcal{O}\left(\frac{1}{z^3}\right). \tag{2.15}$$

Thus we may call G_0 the vector magnetic charge. According to the generalised Dirac quantisation condition [5], [6], the vector magnetic charge has to satisfy an integrality condition. This is easily expressed after rotating G_0 into the CSA, thus obtaining a Lie-algebra element which we write in terms a two-component vector $\vec{g}$ as $i\vec{g}\cdot\vec{H}$ (the vector $\vec{g}$ is not, in general, uniquely defined, but this does not matter here). In the case of minimal symmetry breaking this rotation can always be effected by the action of the unbroken gauge group, and in the case of maximal symmetry breaking G_0 is automatically in the CSA by virtue of the vanishing of the commutator $[\Phi_0, G_0]$. The generalised Dirac condition is the requirement that $\vec{g}$ lies in the dual root lattice, which is spanned by the vectors

$$\vec{\beta}_1^* = \frac{\vec{\beta}_1}{\vec{\beta}_1\cdot\vec{\beta}_1} \quad \text{and} \quad \vec{\beta}_2^* = \frac{\vec{\beta}_2}{\vec{\beta}_2\cdot\vec{\beta}_2}. \tag{2.16}$$

(with our normalisation thus $\vec{\beta}_1^* = \vec{\beta}_1, \vec{\beta}_2^* = \vec{\beta}_2$, but we keep the notational distinction for clarity). According to the Dirac condition there exist integers m_1 and m_2 such that

$$\vec{g} = \frac{4\pi}{e}\left(m_1\vec{\beta}_1^* + m_2\vec{\beta}_2^*\right). \tag{2.17}$$

We will not review the derivation of the Dirac condition here but note that in the physics literature it is usually derived without reference to the symmetry breaking pattern. As we will see in the following section, however, the mathematical status of the two integers m_1 and m_2 depends on the way the gauge symmetry is broken.

3. Configuration spaces and moduli spaces

To clarify the significance of the integers m_1 and m_2 which appear in the Dirac condition we define the (infinite dimensional) space $\mathcal{A}_{\vec{h}}$ as the space or pairs (A_i, Φ) which satisfy the boundary condition (2.13) for some fixed element $\Phi_0 = i\vec{h}\cdot\vec{H}$ of the CSA. This space is not acted on by the group of all static gauge transformations (which is the space of all maps from $\mathbf{R}^3$ to $SU(3)$) but only those whose limit for $z \to \infty$ commutes with Φ_0.

In particular it is therefore acted on by the group of framed gauge transformations $\mathcal{G}_0$, which is defined as

$$\mathcal{G}_0 = \big\{ g : \mathbf{R}^3 \to SU(3) \,\big|\, \lim_{z\to\infty} g(0,0,z) = \mathrm{id} \big\}. \tag{3.1}$$

We may thus define the framed configuration space $\mathcal{C}_{\vec{h}}$ as the quotient $\mathcal{A}_{\vec{h}}/\mathcal{G}_0$. This space is in general not connected but partitioned into disjoint sectors whose labels are called topological charges. The topological charges are elements of the second homotopy group of the quotient of $SU(3)$ by the unbroken gauge group. Thus, since $\Pi_2\big(SU(3)/(U(1)\times U(1))\big) = \mathbf{Z}^2$ and $\Pi_2\big(SU(3)/U(2)\big) = \mathbf{Z}$, the topological charges are pairs of integers in the case of maximal symmetry breaking and a single integer in the case of minimal symmetry breaking. It follows from the results of [13] that in the former case these are the two integers m_1 and m_2 appearing in the expansion (2.17) and in the latter case this is the integer coefficient m_2 of the root $\vec{\beta}_2$ which is not orthogonal to $\vec{h}$.

For minimal symmetry breaking, the vector magnetic charge G_0 is in general not invariant under the action of the unbroken $U(2)$ gauge group and it is therefore not surprising that only the invariant component $\mathrm{tr}(G_0\Phi_0)$ has a topological interpretation. The orbit of G_0 under the action of $U(2)$ is trivial only if G_0 is parallel to Φ_0; otherwise it is a two-sphere in the Lie algebra of $SU(3)$ which we call the magnetic orbit. The magnetic orbit intersects the CSA in two points which are related by a Weyl reflection, and for any configuration (A_i, Φ) in $\mathcal{A}_{\vec{h}}$ these two points again have to satisfy the Dirac condition. Thus, for given topological charge m_2 we can label a magnetic orbit by a pair of integers $[m_1] = \{m_1, m_2 - m_1\}$. The integers $\{m_1, m_2 - m_1\}$ are called magnetic weights in the physics literature and the pair of integers $[m_1]$ is called a holomorphic charge in the mathematics literature. As we mentioned earlier the quantisation of the magnetic weights has a more subtle mathematical origin than the quantisation of the topological magnetic charges. For a precise mathematical discussion we refer the reader to [14]. The important point is that the definition of holomorphic charges requires the connection at infinity as well as the Higgs field. As a result it is possible for the holomorphic charges to change in a continuous deformation of the fields which changes the connection at infinity. Unlike topological charges, holomorphic charges may jump along a path in the configuration space $\mathcal{C}_{\vec{h}}$.

Later we need an explicit parametrisation of the magnetic orbits. Consider the orbit labelled by $([m_1], m_2)$. Assume without loss of generality that m_1 is the larger of the two numbers in $[m_1]$. Then define $\vec{g}$ as in (2.17) and write a general point on the magnetic orbit as

$$G_0 = iP\,\vec{g}\cdot\vec{H}\,P^{-1} \tag{3.2}$$

with P defined as in (2.9). Computing explicitly one finds

$$G_0 = \frac{4m_2\pi i}{e}\left(\frac{\sqrt{3}}{2}H_2\right) + \frac{4\pi i}{e}\mathbf{k}\cdot\mathbf{I}, \tag{3.3}$$

where the vector $\mathbf{k} = (k_1, k_2, k_3)$ has the length $k = |\mathbf{k}| = |m_1 - \frac{m_2}{2}|$ and the direction given by (α, β):

$$\hat{\mathbf{k}} = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta). \tag{3.4}$$

The magnetic orbits play a crucial role in the remainder of this paper and we therefore switch to a labelling which refers directly to their geometry. Formula (3.3) shows that we may picture the magnetic orbits as two-spheres in $su(3)$ with quantised radii $k = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and centres on the one-dimensional lattice $\{\frac{4\pi K i}{e}\left(\frac{\sqrt{3}}{2}H_2\right)|K \in \mathbf{Z}\}$. Thus it is natural to introduce labels (K, k), related to $([m_1], m_2)$ via

$$K = m_2 \quad \text{and} \quad k = |m_1 - \frac{m_2}{2}|. \tag{3.5}$$

For the rest of the paper we will call the vector $\mathbf{k}$ the non-abelian magnetic charge.

To summarise: in the case of maximal symmetry breaking the configuration space $\mathcal{C}_{\vec{h}}$ is partitioned into disjoint topological sectors labelled by the pair of integers (m_1, m_2) but in the case of minimal symmetry breaking the disjoint components of $\mathcal{C}_{\vec{h}}$ are only labelled by one integer K. There exists a finer subdivision according to the magnetic weight but this is not of a topological nature. Instead it is an example of a stratification of a space, with different strata labelled by the magnetic weights. We will describe this concretely at the level of solutions of the Bogomol'nyi equations, or more precisely of certain sets of solutions, called moduli spaces.

Taubes was the first to establish a rigorous existence theorem for monopole solutions in Yang-Mills-Higgs theory for a general gauge group and general symmetry breaking pattern. In [15] he proved the existence of solutions in each of the topological components of the framed configuration space. Applied to our situation this shows that for maximal symmetry breaking monopoles exist for all positive integers m_1 and m_2, but for minimal breaking Taubes' result only implies the existence for given positive topological charge m_2

(the positivity condition results from our choice of sign in the Bogomol'nyi equations; if we had studied the equation with the opposite sign we would obtain anti-monopole solutions with negative topological charges). Taubes' results say nothing about the existence of monopoles with given magnetic weight. On the other hand, a number of explicit solutions have been known for some time. For early papers discussing ansätze for monopole solutions in $SU(3)$ Yang-Mills theory (with or without Higgs field) see [16]. It was also noticed early on that certain solutions of the Bogomol'nyi equations of the critically coupled Yang-Mills-Higgs theory can be obtained by deformations of embedded $SU(2)$ monopole solutions, a possibility first pointed out in [17] and discussed further by Weinberg [12] and Ward [18]. In the latter paper it is shown that in the minimally broken $SU(3)$ theory solutions of arbitrary topological charge m_2 can be obtained by embeddings of $SU(2)$ monopoles but that the magnetic weight is necessarily $[m_1] = [0]$; we will study these solutions in detail below. Bais and Weldon found the first solution which cannot be constructed by embedding an $SU(2)$ solution [19]. This solution has topological charge $m_2 = 2$ and magnetic weight $[m_1] = [1]$; it is spherically symmetric and is part of a six parameter family of axisymmetric solutions later found by Ward [20], which we will discuss in detail later in this paper.

Monopole solutions of the same topological characteristic are conveniently grouped together in moduli spaces. In the case of maximal symmetry breaking there is a canonical way of defining the moduli spaces. For given Φ_0 in the expansion (2.13) we fix the topological labels (m_1, m_2) and consider the set of all solutions of the Bogomol'nyi equation in the corresponding component of the configuration space. In symbols:

$$M^{\max}_{m_1,m_2} = \left\{(A_i, \Phi) \in \mathcal{A}_{\vec{h}} \,\middle|\, D_i\Phi = B_i,\ \vec{g} = \frac{4\pi}{e}\left(m_1\vec{\beta}^*_1 + m_2\vec{\beta}^*_2\right)\right\} /\mathcal{G}_0. \tag{3.6}$$

It follows from Taubes' existence theorem that there is a non-empty moduli space of monopole solutions for each point in the dual root lattice shown in Fig. 1.a with positive coordinates (m_1, m_2). Weinberg [12] long ago counted how many parameters' worth of solutions there are for each point in the dual root lattice. Translated into our language, this determines the dimension of the moduli spaces. Observing carefully the slight differences in conventions, Weinberg's results translate into

$$\dim M^{\max}_{m_1,m_2} = 4(m_1 + m_2). \tag{3.7}$$

As also pointed out by Weinberg, this dimension formula suggests that there exist multi-monopole solutions in this theory which are made up of m_1 well-separated $SU(2)$

monopoles embedded along the root $\vec{\beta}_1$ and m_2 well-separated $SU(2)$ monopoles embedded along the root $\vec{\beta}_2$.

In the case of minimal symmetry breaking only the component of G_0 parallel to Φ_0, labelled by the integer m_2, has a topological significance. Thus one may define the corresponding moduli spaces as

$$M_{m_2} = \left\{ (A_i, \Phi) \in \mathcal{A}_{\vec{h}} \,\middle|\, D_i\Phi = B_i,\ -2\mathrm{tr}(G_0\Phi_0) = \frac{4\pi m_2 i}{e}\vec{\beta}_2\cdot\vec{h} \right\} /\mathcal{G}_0. \tag{3.8}$$

However, as we learnt earlier we may classify finite-energy configurations further according to the magnetic weight. In the mathematical literature it is customary to denote the set of all monopoles in M_{m_2} with magnetic weight $[m_1]$ by $M_{[m_1],m_2}$. Then the moduli spaces are labelled by the same labels as the magnetic orbits, and at the risk of confusing mathematicians who may read this paper we will use our preferred orbit labels (K, k) (3.5) also for the moduli spaces, so we write $M_{K,k}$ instead of $M_{[m_1],m_2}$. As anticipated earlier the spaces $M_{K,k}$ are components of the *connected* space M_K. In mathematical terminology one says that the space M_K is stratified with strata $M_{K,k}$. It was first pointed out by Bowman [21] that counting and interpreting the parameters of monopoles in the case of less than maximal symmetry breaking is considerably more complicated than in the case of maximal symmetry breaking. Murray was the first systematically to compute the dimension of the moduli spaces of solutions in [14]. The situation is summarised in Fig. 1.b.

We can still think of each point of the dual root lattice as representing a (possibly empty) moduli space of solutions, but we need to keep in mind that points related by reflection at the vertical axis (which maps m_1 to $m_2 - m_1$) represent the same moduli space. The results of [14] imply that there only exist monopole solutions if the size of the magnetic orbit is less than or equal to the (positive) topological charge K which means that $k = 0, 1, ..., \frac{K}{2}$ if K is even and $k = \frac{1}{2}, \frac{3}{2}, ..., \frac{K}{2}$ if K is odd. Thus solutions only occur inside the cone (including the edge) drawn in Fig. 1.b, where we also give the dimension which Murray computed for each of the non-empty moduli spaces. Interpreting those dimensions physically is one of the objectives of this paper. To that end, however, we need more explicit descriptions of the moduli spaces. We are particularly interested in the moduli spaces on the edge of the cone and, for even K, in the centre of the cone. For given K these are the strata of respectively smallest and largest dimensions. Hence they are referred to respectively as the small and the large strata.

Before we give more explicit descriptions of the moduli spaces we note that the moduli spaces we have defined are *a priori* only defined as sets. While there may be various mathematically natural ways to give these sets the structure of a manifold, the physically relevant structures are induced from the underlying field theory. In particular one would like to induce the structure of a differentiable manifold from the field theory configuration space and define a Riemannian metric from the field theory kinetic energy functional. This works well for example in the case of $SU(2)$ monopole moduli spaces [22] but is known to be problematic for solitons in the $\mathbf{CP}^1$-model [23]. In the discussion of the moduli spaces $M_{K,k}$ we will also encounter pathologies which are in some sense worse than in the case of $\mathbf{CP}^1$ lumps. However, in order to describe these pathologies explicitly we recall here the formal definition and general form of the metric on the moduli spaces. Consider therefore some generic moduli space M of monopoles and suppose we have introduced (local) coordinates $X = (X_1, ..., X_{\dim M})$ with components X_α, $\alpha = 1, ..., \dim M$ on M. Then a monopole configuration in M may be written as $(A_i, \Phi)(X; \mathbf{r})$, exhibiting explicitly the dependence on both the collective coordinate X and the spatial coordinate $\mathbf{r}$. The metric $g_{\alpha\beta}(X)$ is defined via the L^2 norm of the infinitesimal variations $(\delta_\alpha A_i, \delta_\alpha \Phi)$ which are required to satisfy the linearised Bogomol'nyi equations and, crucially, Gauss' law. In the gauge $A_0 = 0$ this reads

$$D_i \delta_\alpha A_i + [\Phi, \delta_\alpha \Phi] = 0. \tag{3.9}$$

Then the metric can in principle be computed via

$$g_{\alpha\beta}(X) = -\frac{1}{2} \int d^3x \, \mathrm{tr}\, (\delta_\alpha A_i \delta_\beta A_i) + \mathrm{tr}\, (\delta_\alpha \Phi \delta_\beta \Phi) \, . \tag{3.10}$$

Note that by construction the Euclidean group E_3 of translations and rotations in $\mathbf{R}^3$ and the unbroken gauge group act on the moduli spaces we have defined and that the above expression is formally invariant under those group actions. Thus we expect the Euclidean group and the unbroken gauge group to act isometrically on the moduli spaces in all cases where the above metric is well-defined.

4. Embedding SU(2) monopoles

Before we enter the general discussion of the structure of the moduli spaces we note that a large family of $SU(3)$ monopole solutions can be obtained by simply embedding $SU(2)$ monopoles. This family will be particularly important for us. The method is simple: take a simple root which has a positive inner product with Φ_0 and embed the monopole in the associated $SU(2)$ Lie algebra, adding a constant Lie algebra element to satisfy the boundary condition (2.13). In discussing the explicit form of solutions we will take the value of v (2.14) to be $\frac{1}{2\sqrt{3}}$ in order to make contact with other explicit solutions discussed in [24]. Thus, taking the simple root $\vec{\beta}_2$ for definiteness and referring to the definitions (2.7) of the generators U_l, $l = 1, 2, 3$, we define

$$\begin{aligned} \Phi^u &= \sum_{l=1}^{3} \phi_l U_l + \mathrm{diag}(1, -\tfrac{1}{2}, -\tfrac{1}{2}) \\ A_i^u &= \sum_{l=1}^{3} a_{il} U_l, \end{aligned} \tag{4.1}$$

where (a_i, ϕ) is a $SU(2)$ monopole of charge K scaled so that its Higgs field tends to $\mathrm{diag}(\frac{3}{2}, -\frac{3}{2})$ along the positive z-axis. The Higgs field of the embedded solution then has the following expansion along the positive z-axis

$$\Phi^u = \Phi_0 - \frac{K}{ez} U_3 + \mathcal{O}\left(\frac{1}{z^2}\right), \tag{4.2}$$

showing that the embedded solution is an element of the space $M_{K,K/2}$.

Note, however, that this embedding is not unique. We obtain an equally valid solution after acting with the unbroken symmetry group $U(2)$. In terms of the explicitly parametrised element P in (2.9) we define

$$\begin{aligned} \Phi_P &= P \Phi^u P^{-1} \\ A_i^P &= P A_i^u P^{-1}. \end{aligned} \tag{4.3}$$

What is the orbit under the action of P? First note that

$$[Y, U_\pm] = \pm U_\pm \quad \text{and} \quad [I_3, U_\pm] = \mp \frac{1}{2} U_\mp \tag{4.4}$$

and hence that the configuration (4.1) is invariant under the $U(1)$ subgroup generated by $Y + 2I_3$. It follows that the orbit of a given embedded configuration (A_i^u, Φ^u) under the $U(2)$ action is the quotient of $U(2)$ by that $U(1)$ group; this is a three-sphere which we denote by S^3_P. This three-sphere can be coordinatised in a physically meaningful way using the Euler angles (α, β, γ) as defined in (2.9) (the angle χ is redundant). Alternatively we can think of elements of S^3_P as $SU(2)$ matrices Q, given by

$$Q(\alpha, \beta, \gamma) = e^{-\frac{i}{2}\alpha\tau_3} e^{-\frac{i}{2}\beta\tau_2} e^{-\frac{i}{2}\gamma\tau_3} \tag{4.5}$$

This three-sphere is Hopf-fibred over the magnetic two-sphere, defined in (3.2) and we can now write the corresponding Hopf map $\pi^k_{\rm Hopf}$, which depends on the magnitude of the non-abelian magnetic charge:

$$\pi^k_{\rm Hopf} : Q \in S^3_P \to \mathbf{k} \in S^2, \tag{4.6}$$

with $\mathbf{k}$ defined as in (3.4). The angle γ parametrises the 'body-fixed' $U(1)$ rotations about the vector $\mathbf{k}$ (3.3) which only change the monopole's short-range fields. The role of this circle is well-understood in the context of $SU(2)$ monopoles: motion around it gives the monopole electric charge. Thus we will call the circle parametrised by γ the electric circle. Then we can sum up the preceding discussion by saying that the three-sphere S^3_P is Hopf-fibred over the magnetic orbit S^2 with fibre the electric circle.

The embedding procedure just described can be carried out for $SU(2)$ monopoles of arbitrary charge. The moduli space of the latter is well-understood and for magnetic charge K it has the form

$$M_K^{\rm SU2} = \mathbf{R}^3 \times \frac{S^1 \times M_K^0}{\mathbf{Z}_K} \tag{4.7}$$

where $\mathbf{R}^3$ coordinatises the centre-of-mass of the charge k monopoles, the S^1-factor is the electric circle introduced at the end of the previous paragraph and M_K^0 is the K-fold cover of the moduli space of centred (both in $\mathbf{R}^2$ and S^1) $SU(2)$ monopoles of charge K. The space $M_K^{\rm SU2}$ has dimension $4K$. As mentioned earlier, embedding $SU(2)$ monopoles gives rise to $SU(3)$ monopoles of the same topological magnetic charge and with magnetic orbit radius $K/2$. In other words embeddings of $SU(2)$ give families of $SU(3)$ monopoles which are elements of the small strata of the moduli space of $SU(3)$ monopoles. In fact it is easy to see from the Nahm data (see e.g. [21]) that all $SU(3)$ monopoles in the small strata can be obtained via embeddings. Thus putting together our explicit embedding prescription with the formula (4.7) we deduce that the small strata $M_{K,K/2}$ are fibred over the magnetic orbit with the spaces $M_K^{\rm SU2}$ as fibres. We thus have the bijection

$$M_{K,K/2} \leftrightarrow \mathbf{R}^3 \times \frac{S^3_P \times M_K^0}{\mathbf{Z}_K}. \tag{4.8}$$

Here $\mathbf{Z}_K$ acts on S^3_P by moving a fraction $2\pi/K$ round the fibre of this fibration (the electric circle). In the fibration

$$\begin{array}{ccc} M_K^{\rm SU2} & \longrightarrow & M_{K,K/2} \\ & & \big\downarrow \pi^k \\ & & S^2 \end{array} \tag{4.9}$$

the projection map π^k is the forgetful map on $\mathbf{R}^3$ and M_K^0 and the Hopf map (4.6) on S_P^3. For later use we also introduce the notation $M_{K,\mathbf{k}}$ for the fibre $(\pi^k)^{-1}(\mathbf{k})$. Note in particular that the number of independent parameters in the spaces $M_{K,K/2}$ is $4K+2$, thus agreeing with the dimension found for the small strata by Murray.

We have not yet said anything about the differentiable and metric structure which $M_{K,K/2}$ inherits from the field theory kinetic energy functional via (3.10). By inspection one checks that the moduli space metric (3.10) is well-defined on the fibres of the fibration (4.9) and that, with that metric, the fibres are (up to an overall scaling factor of 1/3 required to agree with the conventions of [22]) isometric to the $SU(2)$ monopole moduli spaces. However, the mathematical structure of the magnetic orbit harbours a number of surprises and subtleties, most of them related to physical observations made some time ago and all of them to do with the action of the unbroken gauge group $U(2)$. Since this group action is of central importance for our investigation we discuss it in a separate section.

5. 'Global Colour' revisited

It is a standard lore in the theory of topological defects that if a defect breaks a symmetry of the underlying theory the broken symmetry generators can be used to define collective coordinates for the defect. However it was realised long ago by a number of authors that this is problematic in gauge theories when the gauge symmetry gets broken to a non-abelian group. Historically this discussion was mostly conducted in the context of grand unified theories with gauge group $SU(5)$ broken to $SU(3)_{\mathrm{colour}} \times U(2)_{\mathrm{electroweak}}$ and in that context the question of defining and dynamically exciting the collective coordinates associated to the unbroken gauge group was put succinctly by Abouelsaood : "Are there chromodyons ?" [25].

The answer was given partly by Abouelsaood himself and complemented by the important observation of Nelson and Manohar [26] (see also [27]) that "Global colour is not always defined". Briefly, and applied to our situation, the latter authors noted that if one writes down the Higgs field on the two-sphere at spatial infinity in any regular gauge and attempts to define generators of a $U(2)$ algebra which commute with the Higgs field everywhere and vary smoothly over the two-sphere one will only succeed if the topological magnetic charge of the configuration is even. For odd topological charge there is a topological obstruction similar to the one preventing the existence of a smooth non-vanishing vector fields on a two-sphere; in that case it is only possible to extend a maximal torus of $U(2)$ over the entire two-sphere at infinity. The result of Nelson and Manohar [26] imply that if one insists on defining a $U(2)$ action at a fixed point (say $z = +\infty$) in the case of odd magnetic charge then every extension of it over the entire two-sphere at spatial infinity will necessarily change not only the $1/r$ term in the asymptotic expansion of the Higgs field but even the r^0 part somewhere on the two-sphere at spatial infinity.

However, even if one allows the $U(2)$ action to change the Higgs field (by a gauge transformation) on the two-sphere at spatial infinity there are problems with the collective coordinates produced by the generators which do not commute with the vector magnetic charge G_0. These were pointed out by Abouelsaood [25] who showed that infinitesimal deformations produced by such generators do not satisfy the constraint imposed by Gauss' law (3.9). In the following we will call collective coordinates whose infinitesimal variation produces zero-modes which satisfy Gauss' law and which have a finite L^2-norm (so that the corresponding components of the metric (3.10) are finite) dynamically relevant. Abouelsaood's result is thus that collective coordinates produced by generators of the unbroken gauge group which do no commute with the vector magnetic charge are not dynamically relevant.

How do these observations rhyme with our description of monopole moduli spaces in the two preceding sections? Returning to Fig. 1.b we first note that all moduli spaces on the vertical axis are of the form $M_{K,0}$, with the topological magnetic charge K necessarily even. Moreover the vector magnetic charge G_0 is parallel to the constant part of the Higgs field Φ_0. Thus G_0 is by definition invariant under the action of all generators of the unbroken gauge group and we expect no problems in defining the action of the unbroken gauge group on these spaces. For all the other moduli spaces, however, the magnetic charge G_0 obstructs the action of the unbroken gauge group $U(2)$. It then follows from the results described in the previous paragraph that only the centraliser of G_0 in $U(2)$ can have a dynamically relevant action on these spaces.

The unbroken gauge group $U(2)$ explicitly entered our description of the small strata $M_{K,K/2}$ and it is therefore not difficult to isolate the physically problematic coordinates in those spaces. These are by definition coordinates associated with the generators of the unbroken gauge group which do not commute with G_0 and are therefore precisely the coordinates on the magnetic orbit, i.e. on the base space of the fibration of the spaces $M_{K,K/2}$ in (4.9). Motion on the fibre is physically unproblematic but motion orthogonal to the fibres is physically not permitted. The mathematical reason for this is that the magnetic orbit inherits neither a differentiable nor a metric structure from the field theory. On the other hand, thinking of the magnetic orbit as a two-sphere in the Lie algebra $su(3)$ it is natural, in view of our remarks after (3.4), to induce mathematical structure from this embedding. In the quantum theory, to be discussed in Sect. 10, we will indeed require some mathematical structure on the magnetic orbit, namely an integration measure (which does not presuppose the existence of a metric structure). Thinking of the magnetic orbits as round two-spheres of radius k (this was anticipated in defining the Hopf map π^k_{Hopf} in (4.6)) we thus define the integration measure $k^2 \sin\beta \, d\beta \wedge d\alpha$ on them. Although we have only been able to isolate the magnetic orbit explicitly as part of the moduli space in the smallest strata we expect on physical grounds all strata $M_{K,k}$ with $k > 0$ to be fibred over two-spheres parametrising the non-abelian magnetic charges. This conjecture does not appear to have been considered in the mathematics literature and we are not able to write down the projection maps for these fibrations explicitly. Nontheless we shall assume that projection maps exist for all $M_{K,k}$ provided $k > 0$ (in this paper we will only use them for $k = K/2$) and think of the base spaces of these fibrations as round two-spheres with the integration measure given above.

The observations of Abouelsaood, Nelson and Manohar have lead most authors discussing $SU(3)$ monopoles in the literature to discard the magnetic orbit altogether, see [28] for a recent example. For us there are two reasons for keeping the magnetic orbit in the discussion, and for equipping it with the measure given above. The first is that from a certain mathematical point of view, to be described in Sect. 7, it is very natural to include the magnetic orbit in the moduli spaces. The second and more important reason is that the magnetic orbit plays a crucial role in the full understanding of of (classical and quantised) dyonic excitations and of the behaviour of several interacting monopoles. In particular we will see that is impossible to understand how two quantum states of topological charge one monopoles combine to a quantum state of a topological charge two monopole without taking the magnetic orbit into account. Much of the remainder of this paper is devoted to explaining this point, but as a first step we use the next, short section to exhibit some of the elementary questions which arise when studying classical interacting monopoles with non-abelian magnetic charge.

6. Counting monopoles and their moduli

To begin, focus on the moduli space $M_{1,1/2} = \mathbf{R}^3 \times S^3_P$. Physically this space summarises the degrees of freedom of a single monopole in minimally broken $SU(3)$ Yang-Mills-Higgs theory and is therefore the basic building block for any understanding of monopole physics in that theory. We have seen that of the monopole's six collective coordinates only four are dynamical: three coordinates for the monopole's position and one for the electric circle. Thus, dynamically a single $SU(3)$ monopole has the same degrees of freedom as an $SU(2)$ monopole, but in addition it has a non-dynamical label, namely a point on a two-sphere which specifies the direction of the vector magnetic charge in the Lie algebra of $SU(3)$. The crucial and interesting point is, however, that this magnetic direction and the electric degree of freedom are not independent: the electric circle is generated by the centraliser group of the magnetic charge. Monopoles with different magnetic directions therefore carry charge with respect to different $U(1)$ groups.

Now consider combining two monopoles. This can be done in two distinct ways, corresponding to the two strata of the moduli space M_2 of monopoles of topological charge two. To obtain a configuration in the small stratum $M_{2,1}$ the vector magnetic charges of the individual monopoles have to be parallel but to obtain a configuration in the large stratum $M_{2,0}$ the vector magnetic charges should cancel and thus be anti-parallel. More generally the restriction that the radius k of the magnetic orbit is less than or equal to half the topological magnetic charge K, pictorially expressed in the cone structure of Fig. 1.b, means that at least in principle it is possible to interpret configurations in M_K as being made up of K single monopoles. In particular we know already from our discussion of the small stratum $M_{K,K/2}$ that it indeed contains configurations made up of K monopoles with all their vector magnetic charges aligned. This raises the question of whether all strata have an asymptotic region where the moduli (and in particular their number) can be interpreted in terms of the moduli of individual monopoles. How can this be done? As a general principle we shall assume that in any set of well-separated monopoles which satisfies the Dirac quantisation condition any subset must also satisfy that condition. In particular, therefore, any pair must satisfy the Dirac condition and thus the monopoles' vector magnetic charges must be pairwise parallel or anti-parallel. It follows that in a collection of K monopoles the magnetic vectors of all them necessarily lie along one line, and only the individual positions and electric phases can be chosen independently. One would then expect the dimension of all the strata of the moduli space M_K to be

$4K+2$. In fact this formula is only valid for the smallest strata, where we know the above picture to be correct. Amongst the other strata the strata of smallest magnetic weight for given topological magnetic charge K hint at a completely different interpretation. There the dimension is $6K$, suggesting the physically puzzling interpretation of [14] that in these moduli spaces the individual monopoles' vector magnetic charges have somehow escaped the constraint imposed by the Dirac condition and have become independent and dynamical.

The next sections are devoted to a detailed investigation of these and other questions in the case of the 12-dimensional moduli space $M_{2,0}$. This is the simplest moduli space which cannot be understood via embeddings and as a result we need to consider mathematically more sophisticated approaches.

7. Monopoles and rational maps

The identification of monopole moduli spaces with spaces of rational maps from $\mathbf{CP}^1$ into certain flag manifolds goes back to conjectures of Atiyah and Murray [29] and was first proved in the $SU(2)$ case by Donaldson [30]. Since then generalisations of Donaldson's result to general gauge groups and symmetry breaking have been proved, see e.g. [31] for recent results and further references. The results relevant for us are mostly contained in the papers [14] and [32]. It is explained in [14] that the rational maps describing $SU(3)$ monopoles with minimal symmetry breaking to $U(2)$ are based rational maps from $\mathbf{CP}^1$ to $\mathbf{CP}^2$. Such maps are topologically classified by their degree, and this equals the topological charge of the associated monopole. Concretely the condition that the map is based means that the point at infinity is sent to zero, and we may write such maps as functions of one complex variable $\zeta \in \mathbf{C}$ taking values in $\mathbf{C}^2 \cup \infty$. Then a rational map of degree K has the form

$$R(z) = \left(\frac{p_1(\zeta)}{q(\zeta)}, \frac{p_2(\zeta)}{q(\zeta)} \right), \tag{7.1}$$

where q is a polynomial of degree K whose leading coefficient is 1 and p_1 and p_2 are polynomials of degree less than K. Writing Rat_K for the space of all based rational maps from $\mathbf{CP}^1$ to $\mathbf{CP}^2$ of degree K one of the results of [14] is that there is a one-to-one correspondence between Rat_K and the moduli space M_K (3.8) of monopoles of topological charge K in minimally broken $SU(3)$ Yang-Mills-Higgs theory. There is also a stratification of Rat_K which corresponds to that of the monopole moduli spaces, thus encoding the monopoles' magnetic weights in the rational map. A general account of how this encoding is done can be found in [14]; in the special cases we are concerned with we will be able to identify the magnetic weight explicitly without reference to the general theory. One basic tool for understanding the correspondence between rational maps and monopoles is the action of the symmetry group on the moduli spaces. In our case this is the direct product of the Euclidean group of translations and rotations in $\mathbf{R^3}$ and the unbroken gauge group $U(2)$. This group acts naturally on the moduli space M_K and hence it has an action on Rat_K as well.

The $U(2)$ action on rational maps is easy to write down. Parametrising a $U(2)$ matrix explicitly as $e^{i\chi}Q$, where the $SU(2)$-matrix Q is parametrised as in (4.5) and the angles $(\chi,\alpha,\beta,\gamma)$ satisfy the $\mathbf{Z}_2$ condition specified after (2.9), the $U(2)$ action on the rational map (7.1) is

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \mapsto e^{i\chi}Q \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \tag{7.2}$$

The construction of rational maps from monopoles requires the choice of a preferred direction in $\mathbf{R}^3$ and hence breaks the symmetry of Euclidean space. We follow the conventions of [32] where the preferred direction is the x-direction. Then a translation $(x,y,z)\in\mathbf{R}^3$ acts on a rational map as

$$R(\zeta)\mapsto e^{3x}R(\zeta-\frac{i}{2}(y+iz)). \tag{7.3}$$

The spatial rotation group $SO(3)$ also acts on the rational maps but this action does not concern us here (in fact only the action of the $SO(2)$ subgroup of rotations about the x-axis is known explicitly).

Consider now the simplest case of a monopole of charge one. The associated rational map has the general form

$$R_1(\zeta)=(\frac{\mu_1}{\zeta-\epsilon},\frac{\mu_2}{\zeta-\epsilon}). \tag{7.4}$$

To understand the interpretation of the complex numbers μ_1,μ_2 and $\epsilon=\epsilon_1+i\epsilon_2$ note that his map is obtained from the standard map $(0,1/\zeta)$ by a combined translation (7.3) and a $U(2)$ action if we identify

$$(x,y,z)=\frac{1}{2}(\frac{1}{3}\ln(|\mu_1|^2+|\mu_2|^2),-\epsilon_2,\epsilon_1) \tag{7.5}$$

and

$$\frac{1}{\sqrt{(|\mu_1|^2+|\mu_2|^2)}}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}=\begin{pmatrix} -e^{i(\chi+\frac{1}{2}\gamma-\frac{1}{2}\alpha)}\sin\beta \\ e^{i(\chi+\frac{1}{2}\gamma+\frac{1}{2}\alpha)}\cos\beta \end{pmatrix}. \tag{7.6}$$

Note in particular that the right hand side depends on (χ,γ) only in the combination $(\chi+\frac{1}{2}\gamma)$ and that we can extract the polar coordinates (α,β) for the direction of the non-abelian magnetic charge (3.4):

$$e^{-i\alpha}\tan\beta=\frac{\mu_1}{\mu_2}. \tag{7.7}$$

We thus have the following interpretation of the parameters μ_1,μ_2 and ϵ in terms of monopole moduli. The monopole's position in the yz-plane is given by the complex number ϵ and the x-coordinate is determined by the length of the $\mathbf{C}^2$ vector $(\mu_1,\mu_2)^t$. The corresponding unit length vector in S^3 determines the direction of the non-abelian magnetic charge vector via the Hopf projection

$$\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}\to\frac{\mu_1}{\mu_2} \tag{7.8}$$

and the fibre of that projection is the electric circle, parametrised by $\gamma/2+\chi$. Again we discard the redundant angle χ.

The rational maps describing monopoles of charge two have the general form

$$R_2(\zeta)=(\frac{a+b\zeta}{\zeta^2+f\zeta-\epsilon^2},\frac{c+d\zeta}{\zeta^2+f\zeta-\epsilon^2}). \tag{7.9}$$

The set of all such maps form a 12-dimensional manifold, parametrised by the complex numbers a,b,c,d,ϵ,f. We define the strata of this manifold with the aid of the matrix

$$M=\begin{pmatrix}a&b\\ c&d\end{pmatrix}. \tag{7.10}$$

The small stratum of Rat_2 is defined by the condition that for all its elements the determinant of M vanishes. Thus the small stratum is 10-dimensional and Murray showed [14] that there is a one-to-one correspondence between it and the small stratum $M_{2,1}$ of the charge two $SU(3)$ monopole moduli space. Geometrically the condition $\det M=0$ means that the range of the corresponding rational map lies entirely inside some $\mathbf{CP}^1\subset\mathbf{CP}^2$. Since $\mathbf{CP}^2$ is fibred over $\mathbf{CP}^1$ with fibre $\mathbf{CP}^1$ one deduces that the small stratum of Rat_2 is also fibred over $\mathbf{CP}^1$ with each fibre diffeomorphic to the set of rational maps $\mathbf{CP}^1\to\mathbf{CP}^1$ of degree two. Since the latter set is, by Donaldson's theorem, isomorphic to the moduli spaces of charge two $SU(2)$ monopoles we recover the structure (4.9).

Note that the small stratum of Rat_2 naturally has the structure of a complex differentiable manifold whereas in the fibration (4.9) of the monopole moduli space the fibres, but not the base space, inherit a differentiable structure from the field theory. It follows that the bijection between the small stratum of rational maps and the small stratum of the monopole moduli space is not a diffeomorphism.

The large stratum is defined as the set of maps in Rat_2 for which the determinant of M does not vanish. It is twelve dimensional and was shown by Dancer [32] to be diffeomorphic to the big stratum $M_{2,0}$ of the charge two monopole moduli space. Note that the rational map description of the moduli spaces greatly clarifies the relation between the strata. We can now see explicitly that the strata are part of the connected set Rat_2 and that in a precise sense the small stratum is the boundary of the large stratum.

Concentrating now on the large stratum $M_{2,0}$ we would again like to find a physical interpretation of the parameters occurring in (7.9). By acting on a generic rational map of the form (7.9) with a suitable translation in the yz-plane we can set $f = 0$. Now consider those maps for which $|\epsilon|$ is large. Then we may write the map (7.9) in terms of partial fractions as

$$\tilde{R}_2(\zeta) = \left(\frac{\mu_1}{\zeta-\epsilon} + \frac{\nu_1}{\zeta+\epsilon}, \frac{\mu_2}{\zeta-\epsilon} + \frac{\nu_2}{\zeta+\epsilon}\right), \tag{7.11}$$

with the parameters μ_1, μ_2, ν_1 and ν_2 related to a, b, c, d and ϵ via

$$\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\frac{a}{\epsilon}+b\\ \frac{c}{\epsilon}+d\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-\frac{a}{\epsilon}+b\\ -\frac{c}{\epsilon}+d\end{pmatrix}. \tag{7.12}$$

The map (7.11) is clearly the sum of two degree one maps, and it is tempting to interpret it as describing two monopoles located at

$$\frac{1}{2}(\frac{1}{3}\ln 2|\epsilon|(|\mu_1|^2+|\mu_2|^2), -\epsilon_2, \epsilon_1) \quad \text{and} \quad \frac{1}{2}(\frac{1}{3}\ln 2|\epsilon|(|\nu_1|^2+|\nu_2|^2), \epsilon_2, -\epsilon_1) \tag{7.13}$$

with internal orientation given by the vectors (μ_1, μ_2) and (ν_1, ν_2) normalised to lie on the unit sphere S^3 in $\mathbf{C}^2$. In particular this suggests that the individual monopoles' non-abelian magnetic charges have directions (α_1, β_1) and (α_2, β_2) given by

$$e^{-i\alpha_1}\tan\beta_1 = \frac{\mu_1}{\mu_2} \quad \text{and} \quad e^{-i\alpha_2}\tan\beta_2 = \frac{\nu_1}{\nu_2}. \tag{7.14}$$

What is remarkable here is that the magnetic orientations of the individual monopoles appear as independent, unconstraint coordinates in the moduli space. This appears to support Murray's interpretation [14] that, at least in a suitable asymptotic region, $M_{2,0}$ describes two monopoles with six independent dynamical degrees of freedom each. Leaving aside for a moment the difficulties of talking about the individual monopole charges in a multimonopole configuration (we return to this point in Sect. 9) it is clear that two charge one monopoles with arbitrarily oriented vector magnetic charges would combine to a monopole configuration whose vector magnetic charge in general violates the Dirac condition. In the next two sections we will give a number of arguments why the interpretation of the space $M_{2,0}$ in terms of two monopoles with independent dynamical vector magnetic charges is not correct. Instead the dimensionality of $M_{2,0}$ can be understood by taking into account a new dynamical coordinate which only appears when two monopoles are combined into a monopole configuration of topological charge two.

8. Dancer's moduli space

A detailed study of the space $M_{2,0}$, including its Riemannian structure, was carried out by Dancer in a series of papers [24] -[32] from the point of view of Nahm matrices; see also the papers [33] and [34] with Leese, where the classical dynamics of charge two monopoles is studied. For us the description of the isometries of $M_{2,0}$ in [35] is particularly relevant. There it is shown that $M_{2,0}$ is a hyperkähler manifold whose double cover decomposes as a direct product of hyperkähler manifolds such that one has the isometry

$$M_{2,0} = \mathbf{R}^3 \times \frac{S^1 \times \tilde{M}^8}{\mathbf{Z}_2}, \tag{8.1}$$

where $\tilde{M}^8$ is an eight-dimensional irreducible hyperkähler manifold. The translation group acts only on the $\mathbf{R}^3$ part of the above decomposition and the central $U(1)$ subgroup of the unbroken gauge group $U(2)$ acts only on the S^1 factor. Thus we may think of $\tilde{M}^8$ as the space of centred $SU(3)$ monopoles (in analogy with the space M_2^0 in (4.7) for $SU(2)$ monopoles). The manifold $S^1 \times \tilde{M}^8$ is acted on isometrically by $\mathrm{Spin}(3) \times SU(2)$, where $\mathrm{Spin}(3)$ is the double cover of the group $SO(3)$ of spatial rotations and $SU(2)$ is the quotient group of the unbroken gauge group $U(2)$ by its centre $U(1)$. The centre of $SU(2)$ acts trivially but the centre of $\mathrm{Spin}(3)$ acts non-trivially on both $\tilde{M}^8$ and on S^1, the action on the latter being rotation by π.

On the quotient $M^8 = \tilde{M}^8/\mathbf{Z}_2$ the $\mathrm{Spin}(3)$ action descends to an $SO(3)$ action which commutes with the $SU(2)$ action. Thus by quotienting M^8 further by the $SU(2)$ action one obtains a five-dimensional manifold N^5 which is acted on isometrically by $SO(3)$. Physically this space describes monopoles with fixed centre-of-mass, quotiented by the action of the unbroken gauge group. It turns out to have simple geometrical interpretation.

In [35] it is explained that N^5 can be identified with a certain open subset of the space of symmetric traceless 3×3 matrices, with $SO(3)$ acting by conjugation. We shall now show that one can further associate a unique unoriented ellipse or line segment in $\mathbf{R}^3$ to a given traceless symmetric matrix. This point of view is particularly convenient for us because it exhibits very clearly the orbit structure of the $SO(3)$ action.

Given a traceless symmetric matrix, diagonalise it and order the eigenvalues $\lambda_+ \geq \lambda_0 \geq \lambda_-$. In the generic case $\lambda_+ > \lambda_0 > \lambda_-$ call the associated eigenvectors v_+, v_0, v_- respectively and define an unoriented ellipse in the v_+, v_0 plane whose major axis is along v_+ and has length $A = \lambda_+ - \lambda_-$ and whose minor axis is along v_0 with length $B = \lambda_0 - \lambda_-$. When $\lambda_+ > \lambda_0 = \lambda_-$ this degenerates to a line along v_+ with length $A = \lambda_+ - \lambda_-$ and when $\lambda_+ = \lambda_0 > \lambda_-$ it becomes a circle orthogonal to v_- with radius $A = B = \lambda_0 - \lambda_-$. In either case the coincidence of two eigenvalues means that the matrix is invariant under conjugation by some $O(2)$ subgroup of $SO(3)$ and this invariance is reflected in the axial symmetry of the associated figure. Finally in the completely degenerate case $\lambda_+ = \lambda_0 = \lambda_- = 0$ the associated ellipse degenerates to a point, so both the matrix and the associated figure are kept fixed by the $SO(3)$ action.

When one takes the quotient of the space N^5 by the $SO(3)$ action one obtains a set N^2 which is not a manifold because the isotropy group is not the same at all points of N^5. Nonetheless the set N^2 is very interesting to consider: it contains those parameters in $M_{2,0}$ which cannot accounted for by actions of the symmetry group and which therefore may be thought of as irreducible "shape" parameters of the monopoles. In [35] it is shown that

$$N^2 = \{(D, \kappa) : 0 \leq \kappa \leq 1, 0 \leq D < \frac{2}{3}E(\kappa)\}, \tag{8.2}$$

where $E(\kappa)$ is the elliptic integral

$$E(\kappa) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \kappa^2 \sin^2 \theta}}. \tag{8.3}$$

In our description of N^5 in terms of ellipses we also isolated shape parameters, namely the lengths A and B of the major and minor axes. Using the explicit map between Nahm data and traceless symmetric matrices given in [35] we can write down the relation between these and the coordinates (D, κ) for the space N^2:

$$A = \frac{1}{2}D^2 \quad \text{and} \quad B = \frac{1}{2}(1 - \kappa^2)D^2. \tag{8.4}$$

In Fig. 2 we sketch the parametrisation of the space N^2 in terms of A and B. Using further the map between Nahm data and monopoles we can now in principle establish a correspondence between ellipses in $\mathbf{R}^3$ and monopoles. In practice detailed information about the monopole fields is only available for particularly symmetric configurations. It is these which we will briefly review.

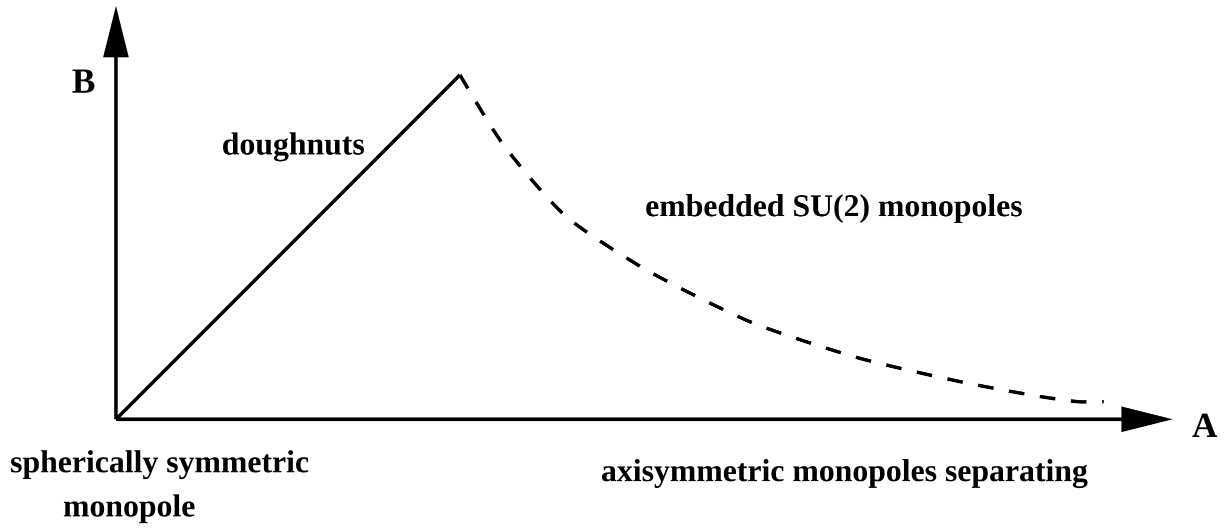

Fig. 2

Monopole configurations parametrised by N^2

The point $A = B = 0$ where the corresponding ellipse becomes a point represents the spherically symmetric monopole (in the sense that a spatial $SO(3)$ rotation acting on such a monopole is equivalent to an internal $SU(2)$ transformation). The precise functional form of such spherically symmetric monopoles was first studied in [19]. The line defined by $\kappa = 0$ corresponds, in the ellipse picture, to a family of circles with radius $0 \leq A = B < 2E(0)/3 = \pi/3$ and represents axisymmetric monopoles, called trigonometric axisymmetric in [35]. They are in fact the family of solutions found by Ward in [20] and have the energy concentrated in a doughnut-shaped region, with the maximum of the energy density on a circle. This family approaches the embedded unique axisymmetric charge two $SU(2)$ monopole as $D \to \pi/3$. As we know from previous sections that embedded solution is an element of the small stratum $M_{2,1}$. More generally, for each $0 \leq \kappa < 1$, configurations in N^2 approach embedded $SU(2)$ monopoles belonging to $M_{2,1}$ in the limit $D \to E(\kappa)$. The condition $\kappa = 1$ also defines a line in N^2 which corresponds to degenerated ellipses of vanishing minor axis B and arbitrary length of the major axis A (for $\kappa \to 1$ the integral defining $E(\kappa)$ diverges logarithmically). This line also represents axisymmetric monopoles, called hyperbolic axisymmetric in [35]. The functional form and energy distribution of these monopoles along the axis of symmetry is given in [24]. Essentially, the family of hyperbolic axisymmetric monopoles interpolates between the spherically symmetric charge two monopole and configurations consisting of two well-separated charge one monopoles, with the separation approximately given by D.

Unfortunately, little is known about the fields of axisymmetric monopoles off the axis of symmetry and about the monopoles represented by a generic point in N^2 (see, however, [33] and [34] for numerical information). The study of the axisymmetric solutions suggests that one of the parameters in N^2 should be thought of as a separation parameter, and that, in the ellipse parametrisation, that separation is given in terms of the major axis as $D = \sqrt{2A}$. Then there is a complementary parameter — in the ellipse picture we think of the length of the minor axis — which parametrises some kind of internal deformation of the monopoles. This parameter corresponds to what in the recent literature [36] has been called the 'non-abelian cloud' of the monopole. It is the goal of the next section to make that notion more precise.

9. Monopoles and non-abelian dipoles

It is known [37] that charge two monopole solutions in spontaneously broken $SU(2)$ Yang-Mills-Higgs theory have no magnetic dipole moments. This is easy to understand when the two monopoles are separated: the charges are equal, and reflection symmetry forces dipole moments relative to the centre-of-mass to be zero. The monopoles in minimally broken $SU(3)$ gauge theory, however, carry vector magnetic charges and two monopoles carrying the same topological magnetic charge may carry different non-abelian magnetic charges. It is then natural to expect multi-monopole configurations made up of two or more such monopoles to have non-vanishing non-abelian magnetic dipole moments. What is more, in view of the difficulties of assigning, even in principle, individual vector magnetic charges to monopoles in a multi-monopole configuration, dipole moments (and possibly higher multipole moments) appear to be the only source of information about the magnetic charges of the individual monopoles. This is the point of view we will adopt in this section.

All the fields we study here have Higgs fields whose asymptotic expansion is consistent with the general form

$$\Phi(\mathbf{r}) = \Phi_0(\hat{\mathbf{r}}) - \frac{G_0(\hat{\mathbf{r}})}{4\pi r} + \frac{id^{aj}\, I_a(\hat{\mathbf{r}})\, \hat{r}_j}{e^2 r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \tag{9.1}$$

with summation on repeated indices. Here $\Phi_0(\hat{\mathbf{r}})$ and $G_0(\hat{\mathbf{r}})$ are the vacuum expectation value of the Higgs field and the vector magnetic charge on the two-sphere at infinity in some gauge (for configurations in $M_{2,0}$, $\Phi_0(\hat{\mathbf{r}})$ and $G_0(\hat{\mathbf{r}})$ are parallel everywhere), and $I_a(\hat{\mathbf{r}})$, $a = 1, 2, 3$, are the generators of the unbroken $SU(2)$ gauge group at a point $\hat{\mathbf{r}}$ on the two-sphere at infinity (since K is even there is no topological obstruction to writing them down on the entire two-sphere at infinity). Without entering a general discussion of multipole expansions of non-abelian gauge fields we interpret the $(1/r^2)$-terms in the above expansion as a dipole term. The Bogomol'nyi equation relates the $(1/r^2)$-terms in the Higgs fields with $(1/r^3)$-terms in the magnetic field and we may thus consider the coefficient matrix d^{aj}, $a, j = 1, 2, 3$, which transforms as a vector both under spatial rotations and under rigid $SU(2)$ gauge transformations, as a non-abelian magnetic dipole moment.

We are interested in the dipole moments of monopole configurations in $M_{2,0}$, but unfortunately, explicit computations of the Higgs field for a generic configuration in this space have only been carried by a numerical implementation of the Nahm transformation (see [33] and [34]). The direct extraction of the dipole moments from the (explicitly known) Nahm data seems very difficult so we restrict our discussion to configurations with extra symmetries or to certain limits, where analytic expressions for the Higgs field have been given in the literature. The configurations we shall discuss are essentially those which are on or near the boundary of the set N^2 depicted in Fig. 2.

We begin with hyperbolic axisymmetric monopoles. The Higgs field on the axis of symmetry, taken to be the z-axis, was written down in [24], and extracting the $(1/z^2)$-term we indeed find a non-abelian component:

$$\Phi(0,0,z) = \left(1 - \frac{1}{2ez}\right)\Phi_0 + \frac{iD\coth 3D}{2e^2 z^2}\, I_3 + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{9.2}$$

Since we do not know the Higgs field along any ray other than the z-axis we cannot deduce the full tensorial structure of the dipole moment. However, we conjecture that for a hyperbolic axisymmetric monopole configuration consisting of two monopoles well-separated along the z-axis the dipole moment is of the form $d^{aj} = d\delta^{a3}\delta^{j3}$. From (9.2) the coefficient d is then $D \coth 3D/2$. For large D, $D \coth D \approx D$, so the dipole moment increases linearly with separation. This is precisely the dipole moment one expects for a configuration containing two monopoles, separated along the z-axis, with equal and opposite non-abelian magnetic charges.

In the limit $D \to 0$ the monopoles coalesce to the spherically symmetric solution. Remarkably the dipole moment does not vanish in this limit. Exploiting the spherical symmetry we can now write down the asymptotic form of the solution everywhere. In the so-called singular gauge, where the gauge-potential has a Dirac-string singularity, it has the form

$$\Phi(\mathbf{r}) = \left(1 - \frac{1}{2er}\right)\Phi_0 + \frac{i}{6e^2r^2}\,\mathbf{I}\cdot\hat{\mathbf{r}} + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{9.3}$$

The non-abelian dipole moment is thus seen to be of the hedgehog form $d^{ai} = d\delta^{ai}$: the Lie algebra components of the dipole are correlated with their spatial direction in such a way that the non-abelian dipole moment is invariant under simultaneous spatial rotations and global $SU(2)$ gauge rotations. In particular the dipole moment therefore does not single out a preferred spatial direction.

Moving on to trigonometric axisymmetric monopoles we find the following expansion along the axis of symmetry, taken to be the z-axis:

$$\Phi(0,0,z) = i\left(1 - \frac{1}{2ez}\right)\Phi_0 + \frac{iD\cot 3D}{2e^2z^2}\,I_3 + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{9.4}$$

Again we are unable to deduce the tensorial structure of the non-abelian dipole moment, but we can determine the component $d^{33} = \cot 3D/2$. Since all trigonometric axisymmetric monopoles have their energy density concentrated in a finite region of space, their dipole moments cannot be understood in terms of the separation and individual non-abelian monopole charges of two single monopoles. Rather, the above expansion shows that trigonometric axisymmetric monopoles have an intrinsic non-abelian dipole moment. In the limit $D \to \pi/3$, where the monopole configurations approach the toroidal charge two monopole in the small stratum, the dipole strength goes to infinity, giving a very physical interpretation of the transition between the strata: the infinite non-abelian dipole moment combines with the (essentially abelian) vector magnetic charge of the large stratum $\mathrm{diag}(\frac{1}{2}, \frac{1}{2}, -1)$ to produce the vector magnetic charge $\mathrm{diag}(0, 1, -1)$ characteristic of the small stratum.

Finally we turn to the asymptotic form of configurations near the boundary of N^2 drawn as a dashed line in Fig. 2. This is studied in [38] where it is argued that for these monopole configurations the term in the Higgs field which we call the dipole term is again of the hedgehog form:

$$\Phi(\mathbf{r}) = i\left(1 - \frac{1}{2ez}\right)\Phi_0 - \frac{iD}{2(\pi - 3D)e^2z^2}\,\mathbf{I}\cdot\hat{\mathbf{r}} + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{9.5}$$

Note that near the boundary the coefficient$D/2(\pi - 3D)$ is necessarily large and tends to infinity as $D \to \pi/3$.

Based on the sample of configurations studied here we propose that quite generally the parameters in N^2 can be understood physically as characterising the monopoles' non-abelian dipole moments. Configurations on the line $\kappa = 1$ which consist of two well-separated monopoles have a purely extrinsic dipole moment. This results from the individual monopoles' opposite non-abelian magnetic charges and is essentially a measure of the monopole separation. Configurations away from the line $\kappa = 1$ have dipole moments which cannot be understood in terms of the individual magnetic charges. In particular the fact that configurations made up of two well-separated monopoles have dipole moments of the hedgehog type (9.5) in a certain limit shows that the fields of the two monopoles which make up such configurations must be quite different from the fields of single monopoles in $M_{1,1/2}$. The asymptotic form (9.5) rules out the possibility, suggested by the rational map description, that in $M_{2,0}$ two well-separated monopoles have magnetic charges with independent directions. Instead it suggests that such monopoles still have their magnetic charges anti-aligned, but that in addition they have individual dipole moments of the hedgehog type. Thus we propose that the elusive cloud parameter is a measure of the individual monopoles' dipole strength. In the limit $D \to E(\kappa)$ this dipole strength tends to infinity. The total dipole moments is then also of the hedgehog type since, for fixed separation, the extrinsic dipole moment is negligibly small relative to the individual hedgehog dipole moments in this limit.

Armed with a physical interpretation of the moduli in $M_{2,0}$ we now turn to the quantisation of monopoles. There, the conclusion that in any configuration in $M_{2,0}$ which consists of well-separated monopoles the individual monopoles' non-abelian charges are anti-parallel will be crucial.

10. Monopole quantisation, the dyon spectrum and the emergence of $U(2) \ltimes \mathbf{R}^4$

The quantisation of BPS monopoles in Yang-Mills-Higgs theory with gauge group $SU(2)$ has been studied extensively in the literature. In that case the moduli spaces are smooth manifolds with finite metrics (and hence well-defined integration measure) and in the standard bosonic quantisation scheme one takes the Hilbert space of states to be the set of all square-integrable functions on the moduli space, see e.g. [39] and [40]. This quantisation scheme needs to be amended before it can be applied to the $SU(3)$ monopoles studied here: the smallest strata of the moduli spaces, for example, are fibred over a two-sphere (4.9) (the magnetic orbit) which inherits from the field theory neither a metric nor a measure nor even a differentiable structure. The largest strata (for even K), on the other hand, are smooth manifolds with finite metrics and the above scheme can again be applied without difficulty. Our prescription will have to take these differences into account.

Our basic starting point is that magnetic charges label superselection sectors in quantised Yang-Mills-Higgs theory. The rationale here is that magnetic charges are conserved independently of the equations of motion and that there are no finite-action configurations which interpolate between classical configurations with different magnetic charges. Thus there can also be no quantum mechanical transitions between states with different magnetic charges. This superselection rule is implicit in the standard quantisation scheme applied to monopoles in theories with abelian unbroken gauge group. There all the magnetic charges are topological and label disjoint moduli spaces. Superpositions between states of different magnetic charges would correspond to linear combinations of wavefunctions on different moduli spaces, and these are usually ruled out. In our case, the superselection sectors are labelled by pairs $(K, \mathbf{k})$ of topological and non-abelian magnetic charges. If K is even and the non-abelian magnetic charge vanishes the corresponding sector takes the familiar form: it is given by the Hilbert space of square-integrable functions on the moduli space $M_{K,0}$ which, being the largest stratum for given K, is equipped with a smooth and finite metric. In symbols

$$\mathcal{H}_{K,0} = L^2(M_{K,0}). \tag{10.1}$$

If $k > 0$ on the other hand, we label sectors also by a point $\mathbf{k}$ on the magnetic orbit and define it to be the Hilbert space of square-integrable functions on the corresponding fibre $M_{K,\mathbf{k}}$ defined after (4.9) for the case $k = K/2$ (as also noted there, however, we expect no principal difficulty in defining the fibres $M_{K,\mathbf{k}}$ for any of the allowed pairs (K, k)). Since these fibres are again equipped with finite and smooth metrics this definition makes sense. However, there is another way to define the superselection sectors labelled by $(K, \mathbf{k})$ which is more useful for us. The starting point here is interpretation in the second but last paragraph of Sect. 5 of the base spaces of the fibration $M_{K,k} \to S^2$ as round spheres with integration measure $k^2 \sin\beta d\beta \wedge d\alpha$. This allows us to define a measure on $M_{K,k}$, namely the product measure of $k^2 \sin\beta d\beta \wedge d\alpha$ with the natural measure on the fibres (the one coming from the metric induced by the field theory). Then it makes sense to define the Hilbert space

$$\mathcal{H}_{K,k} = L^2(M_{K,k}). \tag{10.2}$$

From this larger space we can project out the desired superselection sectors as follows. Consider the operator

$$p_i : \mathcal{H}_{K,k} \to \mathcal{H}_{K,k} \tag{10.3}$$

whose action on a function $\phi \in \mathcal{H}_{K,k}$ is

$$p_i \circ \phi(X) = k_i \phi(X), \tag{10.4}$$

with $\mathbf{k} = \pi^k(X)$ in terms of the projection map π^k defined in (4.9) for $k = K/2$ but conjectured to exist for all $k > 0$ (in the allowed range) in Sect. 5. The simultaneous eigenspaces $\mathcal{H}_{K,\mathbf{k}}$ of the three operators (p_1, p_2, p_3) with eigenvalues (k_1, k_2, k_3) contain precisely the states with definite magnetic charge $\mathbf{k} = (k_1, k_2, k_3)$ and thus correspond to the required superselection sectors. As we shall see, this construction has a very natural interpretation later in this paper.

Having defined the Hilbert spaces for our quantisation scheme we turn to the actions of the various symmetry groups on these spaces. Generally speaking the symmetry group of the theory acts on the moduli spaces, and hence also on functions on the moduli spaces, which can therefore be organised into representations of the symmetry group. In our case the spatial symmetry groups of translations and rotations act smoothly on the moduli spaces, and wavefunctions can correspondingly be organised into momentum and angular momentum eigenstates. Here we are only interested in the transformation properties of of wavefunctions under the action of the unbroken gauge group $U(2)$. As explained in Sect. 5 this action is smooth and isometric only on the largest stratum of moduli spaces for monopoles with even topological charge. On all other strata it is obstructed by the non-abelian magnetic charge so that only the centraliser of the vector magnetic charge acts smoothly and isometrically. Our superselection sectors reflect that difference. Thus we expect to be able to organise the elements of $\mathcal{H}_{K,0}$ into representations of $U(2)$ while elements of $\mathcal{H}_{K,\mathbf{k}}$ should be organised into representation of the centraliser of $\mathbf{k}$. In the former case we may therefore denote dyonic quantum states by $|K, 0; N, j, m\rangle$, where K

is the topological magnetic charge, N is an integer which specifies the representation of the $U(1)$ transformation generated by Y (2.8) and the half-integers $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}...\}$, $m \in \{-j, -j+1, ..., j-1, j\}$ specify a state $|jm\rangle$ in an $SU(2)$ representation; the three numbers (Njm) then specify a state in an $U(2)$ representation, and we need to impose the $\mathbf{Z}_2$ condition that N is odd if j is a half-odd integer and N is even if j is an integer. The other sort of dyonic states can be written as $|K, \mathbf{k}; N, s\rangle$, where K and N have the same meaning as before, $\mathbf{k}$ is the non-abelian magnetic charge defined in (3.4), and s is a single half-integer which specifies the representation of the $U(1)$ subgroup of $SU(2)$ which leaves $\mathbf{k}$ fixed. Again we have the condition that N is odd if s is half-odd and N is even if s is integer.

We now show how these states are realised as wavefunction on the moduli spaces. Since all the essential features of the scheme we are going to propose show up already for monopoles of topological charge $K \leq 2$, we will restrict attention to the corresponding moduli spaces in the following. Thus we will be able to draw on the explicit description of those spaces in the previous sections.

Again we begin with a single monopole and the moduli space $M_{1,1/2} = \mathbf{R}^3 \times S^3_P$. As explained in sects. 5 and 6, this space is fibred over the magnetic orbit S^2. A point $\mathbf{k}$ on the magnetic orbit specifies the non-abelian magnetic charge and obstructs the $U(2)$ action: only the centraliser acts smoothly. For an explicit description of quantum states of a single monopole it is convenient to write a generic $SU(2)$ matrix in terms of Pauli matrices and Euler angles (α, β, γ) as in (4.5) and to introduce Wigner functions $D^j_{ms}(Q) = e^{-im\alpha} d^j_{ms}(\beta) e^{-is\gamma}$ on $SU(2)$ following the conventions of [41]. The functions D^j_{ms}, $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, ...\}$, $m, s \in \{-j, -j+1, ..., j-1, j\}$ form a basis for the Hilbert space $L^2(SU(2))$, and one can moreover (formally) expand the δ-function on $SU(2)$ in terms of them. We write the δ-function peaked at the point Q and with argument Q' as $\delta_E(Q'Q^{-1})$, where E is the identity element. Then, putting primes on the Euler angles for Q' we have the formula

$$\delta_E(Q'Q^{-1}) = \delta(\alpha' - \alpha)\delta(\cos\beta' - \cos\beta)\delta(\gamma' - \gamma) \tag{10.5}$$

and the expansion

$$\delta_E(Q'Q^{-1}) = \sum_{j=0,\frac{1}{2},1...} \sum_{m=-j}^{j} \sum_{s=-j}^{j} \frac{2j+1}{16\pi^2} D^{*j}_{ms}(Q') D^j_{ms}(Q). \tag{10.6}$$

Quantum states for a single monopole are of the form $|1, \mathbf{k}; N, s\rangle$, where $|\mathbf{k}| = 1/2$

and we have the constraint $N+2s=0$ as a consequence of the monopole's invariance under the generator $Y+2I_3$. Since the point $\mathbf{k}$ on the magnetic orbit is obtained from a general point Q on $SU(2)$ via the Hopf projection we have the explicit parametrisation in terms of Euler angles $\hat{\mathbf{k}}=(\sin\beta\cos\alpha,\sin\beta\sin\alpha,\cos\beta)$. Furthermore we can give an explicit realisation of the states of a single monopole in terms of Euler angles:

$$\langle\chi',\alpha',\beta',\gamma'|1,\mathbf{k};N,s\rangle=e^{iN\chi'}\delta(\alpha'-\alpha)\delta(\cos\beta'-\cos\beta)e^{is\gamma'}, \tag{10.7}$$

where the condition $N=-2s$ ensures that the right hand side only depends on $2\chi'-\gamma'$. Then, using the above completeness relation (10.6) we also deduce

$$\langle\chi',\alpha',\beta',\gamma'|1,\mathbf{k};N,s\rangle=e^{iN\chi'}\sum_{j\geq|s|}\sum_{m=-j}^{m=j}\frac{2j+1}{4\pi}D^{j*}_{ms}(Q')e^{-im\alpha}d^j_{ms}(\beta). \tag{10.8}$$

Turning next to two monopoles, we have to distinguish the two strata. Quantum states on the small stratum are again much like the quantum states of a single monopole. They are of the form $|2,\mathbf{k};N,s\rangle$, where now $|\mathbf{k}|=1$ and we again have the constraint $N+2s=0$. These states can be realised as wavefunctions on the moduli space $M_{2,1}$. The most general wavefunction will also depend on the centre-of-mass position and on the relative coordinates summarised in the Atiyah-Hitchin manifold M_2^0, but here we are only interested in the dependence on S^3_P. This dependence is just like for the quantum states of a single monopole:

$$\langle\chi',\alpha',\beta',\gamma'|2,\mathbf{k};N,s\rangle=e^{iN\chi'}\sum_{j\geq|s|}\sum_{m=-j}^{m=j}\frac{2j+1}{4\pi}D^{j*}_{ms}(Q')e^{-im\alpha}d^j_{ms}(\beta), \tag{10.9}$$

where now $|\mathbf{k}|=1$. There may appear to be an inconsistency in our notation at this stage, in that the magnetic labels appear on the right hand side of (10.8) and (10.9), but not on the left. However, the rationale for our notation will soon become clear.

The large stratum $M_{2,0}$ is smooth with a finite metric and, like in the case of $SU(2)$ monopoles, quantum states can be realised as smooth wavefunctions on the moduli space. The magnetic orbit is trivial in this case, so all coordinates should be treated on the same footing. The most general wavefunction depends on the $U(2)$ Euler angles $(\chi,\alpha,\beta,\gamma)$, on the centre-of-mass position, on spatial Euler angles and on the shape parameters (A,B). A detailed investigation of the quantum mechanics on $M_{2,0}$ would be a very interesting but also very challenging project. Here we are merely interested in the transformation properties of the wavefunction under the unbroken gauge group $U(2)$ so we focus again on the wavefunction's dependence on the corresponding coordinates. Making use of the fact that, for any fixed s in the allowed range, both the Wigner functions D^j_{ms} and their complex-conjugates D^{j*}_{ms}, $m \in \{-j, -j+1, ..., j-1, j\}$ span the spin j representation of $SU(2)$, we may choose a value for s and then represent dyonic quantum states as follows:

$$\langle \chi', \alpha', \beta', \gamma' | 2, 0; N, j, m\rangle_s = \frac{\sqrt{2j+1}}{4\pi} e^{iN\chi'} D^{j*}_{ms}(Q'). \tag{10.10}$$

Different values for s lead to equally valid realisations of the state $|2, 0; N, j, m\rangle$, but we introduce the suffix s here because we shall see further below that this half-integer also has a physical interpretation.

Having represented dyonic quantum states with topological magnetic charges one and two as wavefunctions on the relevant moduli spaces we are in a position to address one of the key concerns of this paper: what is the relationship between the tensor product of two quantum states of a single monopole and a quantum state of a topological charge two monopole? The answer to this question can in principle be deduced from a knowledge of the algebraic object whose representations are being studied. In our case we do not yet know the relevant algebraic object. It is clearly not simply the unbroken gauge group $U(2)$ since the quantum states we have studied carry in general only a representation of a subgroup of it, namely the centraliser of the non-abelian magnetic charge. The algebraic object we seek should incorporate the subtle interplay between magnetic and electric properties which we have seen in the dyonic states discussed so far. We will now argue that for dyonic states in the present theory the relevant algebraic object is the semi-direct product $U(2) \ltimes \mathbf{R}^4$. Denoting the generators of $U(2)$ again by I_1, I_2, I_3 and Y and calling the translation generators p_1, p_2, p_3 and P the Lie algebra of $U(2) \ltimes \mathbf{R}^4$ has the following commutation relations

$$\begin{gathered} [I_a, I_b] = i\epsilon_{abc} I_c \\ [p_a, p_b] = 0 \\ [I_a, p_b] = i\epsilon_{abc} p_c \\ [Y, P] = [Y, I_a] = [Y, p_a] = [P, I_a] = [P, p_a] = 0, \text{ for } a = 1, 2, 3. \end{gathered} \tag{10.11}$$

To test this proposal we first show that the representations contain precisely the dyonic states discussed above and then show that the Clebsch-Gordan coefficients of $U(2) \ltimes \mathbf{R}^4$ are consistent with the representation of states as wavefunctions on moduli spaces.

The semi-direct product $U(2) \ltimes \mathbf{R}^4$ is an example of a regular semi-direct product and therefore its representation theory is most effectively dealt with via the method of induced representations [42]. For a general regular semi-direct product $S = H \ltimes N$, where N is an abelian, normal subgroup of S, the construction of an irreducible representation begins with the classification of all H orbits O in $\hat{N}$, the set of characters of N. Each orbit has a characteristic centraliser group C (the centraliser group of any point on it) and a unitary irreducible representations (UIR) can then be induced from the group $C \ltimes N$. The representation space $V_{O,\rho}$ of an UIR of S constructed in this way is then labelled by the orbit O and a UIR ρ of the stability group C:

$$\begin{aligned} V_{O,\rho} = \big\{\phi : H \to V_\rho \big| \phi(QX) =& \rho(X^{-1})\phi(Q) \ \forall Q \in H, X \in C \\ & \text{and} \int_{H/C} ||\phi||^2(z) d\mu(z) < \infty \big\}, \end{aligned} \tag{10.12}$$

where V_ρ is the carrier space of the representation ρ, $||\cdot||$ is the norm induced by the inner product in V_ρ and $d\mu$ is an invariant measure on the coset $H/C \cong O$. One can show that all UIR's of S can be obtained in this way.

Applying this theory to our case we first note an obvious simplification. Since $U(2) = \big(U(1) \times SU(2)\big)/\mathbf{Z}_2$ we have a corresponding direct product decomposition $U(2) \ltimes \mathbf{R}^4 = \big((U(1) \ltimes \mathbf{R}) \times (SU(2) \ltimes \mathbf{R}^3)\big)/\mathbf{Z}_2$. The interesting part of the representation theory comes from the non-abelian part $SU(2) \ltimes \mathbf{R}^3$, which happens to be the double cover of the Euclidean group in three dimensions and whose representation theory is particularly well documented in the literature. In this case the set of characters $\hat{\mathbf{R}}^3$ of the translation group $\mathbf{R}^3$ is isomorphic to $\mathbf{R}^3$ and can physically be thought of as momentum space, with elements denoted by $\mathbf{k}$. The $SU(2)$ orbits in $\hat{\mathbf{R}}^3$ are spheres with radius $k > 0$ or simply a point. In the former case the centraliser group is $U(1)$ and in the latter the entire group $SU(2)$. By the general theory we thus obtain two sorts of UIR's of $SU(2) \ltimes \mathbf{R}^3$. If $k > 0$ the carrier spaces are infinite dimensional and labelled by the orbit size $k > 0$ and a half-integer s specifying a $U(1)$ representation. They may be realised as follows

$$\begin{aligned} V_{k,s} = \big\{\phi : SU(2) \to \mathbf{C} \,\big|\, \phi(Qe^{-\frac{i}{2}\xi\tau_3}) =& e^{is\xi}\phi(Q) \\ & \text{and} \int_{S^2} |\phi|^2(\alpha,\beta) \sin\beta \, d\beta d\alpha < \infty \big\}. \end{aligned} \tag{10.13}$$

The action of an element $(A, \mathbf{a}) \in SU(2) \ltimes \mathbf{R}^3$ on this Hilbert space is given by

$$((A, \mathbf{a}) \circ \phi)(Q) = e^{i\mathbf{ak}}\phi(A^{-1}Q), \tag{10.14}$$

where $\mathbf{k}$ is obtained from Q via the Hopf projection (4.6) i.e. $\mathbf{k} = \pi^k_{\mathrm{Hopf}}(Q)$. Having earlier introduced Wigner functions $SU(2)$ as a basis of $L^2(SU(2))$ one checks that for fixed s the functions D^{j*}_{ms}, $j \in \{|s|, |s|+1, ...\}$,$m \in \{-j, -j+1, ..., j-1, j\}$ form a basis of $V_{k,s}$.

In the case $k = 0$ the general prescription of induced representation theory starts from a standard $(2j+1)$-dimensional representation of $SU(2)$ with carrier space $\mathbf{C}^{2j+1}$ on which $Q \in SU(2)$ is represented by the matrix $D^j_{ms}(Q)$, $s, m \in \{-j, -j+1, ..., j-1, j\}$, and leads to the Hilbert space

$$V_{0,j} = \{\phi : SU(2) \to \mathbf{C}^{2j+1} | \phi_s(QQ') = \sum_{t=-j}^{j} D^j_{st}(Q'^{-1})\phi_t(Q)\}, \tag{10.15}$$

where ϕ_s denotes the s-th component of ϕ. The action of $(A, \mathbf{a}) \in SU(2) \ltimes \mathbf{R}^3$ reduces to an $SU(2)$ action:

$$((A, \mathbf{a}) \circ \phi)(Q) = \phi(A^{-1}Q), \tag{10.16}$$

Again it is straightforward to write down a basis. Since the right action of $SU(2)$ on itself is transitive the value of any element $\phi \in V_{0,j}$ is determined by its value at the identity. Since the possible values at the identity are parametrised by $\mathbf{C}^{2j+1}$ we see explicitly that $V_{0,j}$ is $(2j+1)$-dimensional. We further define a basis $\{\phi^{(m)}\}_{m=-j,-j+1,...,j-1,j}$ consisting of those elements of $V_{0,j}$ which reduce to the canonical basis of $\mathbf{C}^{2j+1}$ at the identity: $\phi^{(m)}_s(E) = \delta_{ms}$. It then follows that

$$\phi^{(m)}_s(Q) = D^j_{sm}(Q^{-1}) = D^{j*}_{ms}(Q). \tag{10.17}$$

By now it will be apparent to the reader that Wigner functions are omnipresent in the representation theory of the Euclidean group, where they play a number of different roles. Here we find them as the components of $\mathbf{C}^{2j+1}$-valued basis functions of $V_{0,j}$. Note in particular that under the $SU(2)$ transformation (10.16) the components of the vector-valued functions $\phi^{(m)}$ do not get mixed:

$$(A \circ \phi^{(m)})_s(Q) = \phi^{(m)}_s(A^{-1}Q) = \sum_{l=-j}^{j} D^j_{lm}(A)\phi^{(l)}_s(Q). \tag{10.18}$$

Thus, for any fixed s in the allowed range the component functions $\phi^{(m)}_s(Q) = D^{j*}_{ms}(Q)$, $m \in \{-j, -j+1, ..., j-1, j\}$ span an equally valid carrier space of the spin j representation of $SU(2)$.

In order to complete our account of the representation theory of $U(2) \ltimes \mathbf{R}^4$, we need to combine the above representations with representations of $U(1) \ltimes \mathbf{R}$. Thinking about the latter from the point of view of induced representation may seem unnecessarily complicated but it is useful for a unified view. The $U(1)$ action on $\mathbf{R}$ (by conjugation) leaves every point fixed, so all orbits are trivial and consist of a real number $K \in \hat{\mathbf{R}}$. The centraliser is always the whole of $U(1)$, so centraliser representations are labelled by a single integer N. All UIR's are one-dimensional and given by

$$v_{K,N} = \big\{\phi : U(1) \to \mathbf{C} \big| \phi(\chi + \xi) = e^{iN\xi}\phi(\chi)\ \forall \chi, \xi \in [0, 2\pi)\big\}. \tag{10.19}$$

Clearly this is just the one-dimensional space spanned by the function $\phi_N(\chi) = e^{iN\chi}$. The action of an element $(\xi, a) \in U(1) \ltimes \mathbf{R}$ on this function is

$$(\xi, a) \circ \phi_N(\chi) = e^{iKa}\phi_N(\chi - \xi). \tag{10.20}$$

The representations of $SU(2) \ltimes \mathbf{R}^3$ and of $U(1) \ltimes \mathbf{R}$ can now be combined to representations of $U(2) \ltimes \mathbf{R}^4$ by taking tensor products of individual representations, but respecting the $\mathbf{Z}_2$ conditions on N and j or N and s outlined earlier. Thus carrier spaces of UIR's of $U(2) \ltimes \mathbf{R}^4$ are of the form

$$V_{K,k;N,s} = v_{K,N} \otimes V_{k,s} \tag{10.21}$$

with $k > 0$ and $N + 2s$ even, or

$$V_{K,0;N,j} = v_{K,N} \otimes V_{0,j} \tag{10.22}$$

with $N + 2j$ even. We will use the unifying notation $V_{K,k;N,n}$ for these representations, with n standing for the $U(1)$ representation label $s \in \frac{1}{2}\mathbf{Z}$ if $k > 0$ for the $SU(2)$ representation label $j \in \{0, \frac{1}{2}, 1, ...\}$ if $k = 0$.

We now come to the promised identification of dyonic quantum states of $SU(3)$ monopoles with elements of $U(2) \ltimes \mathbf{R}^4$ representations, partly already anticipated by our notation. Our proposal is to identify the magnetic charges with the eigenvalues of the translation generators P and p_1, p_2, p_3 of $U(2) \ltimes \mathbf{R}^4$, and the electric charges with the representations of the centraliser subgroups of $U(2)$. More precisely we identify the topological magnetic charge with the representation label K and the magnitude of the non-abelian magnetic charge with the representation label k. Then we associate the dyonic states discussed earlier with the representation spaces of $U(2) \ltimes \mathbf{R}^4$ as follows. For $k > 0$, the realisation of states $|K, \mathbf{k}; N, s\rangle$ as wavefunctions on the moduli space (10.8) and (10.9) shows that they can be written as infinite sums of elements of $V_{K,k;N,s}$. In fact these sums do not converge, but since these states are eigenstates of the translation generators p_1, p_2 and p_3, their non-normalisability is the familiar property of of momentum eigenstates in quantum mechanics on $\mathbf{R}^3$. Keeping that proviso in mind we write

$$|K, \mathbf{k}; N, s\rangle \in V_{K,k;N,s}. \tag{10.23}$$

The dyonic states with $k = 0$ can be interpreted as elements of the representation space $V_{K,0;N,j}$ if we identify the label s in the realisation of $|K, 0; N, j, m\rangle$ (10.10) with the s-th component of elements of $V_{K,0;N,j}$:

$$\langle \chi', \alpha', \beta', \gamma' | K, 0; N, j, m\rangle_s = e^{iN\chi'} \phi_s^{(m)}(Q'). \tag{10.24}$$

With this identification we can then also write

$$|K, 0; N, j, m\rangle \in V_{K,0;N,j}. \tag{10.25}$$

One immediate question which arises after this identification concerns the quantisation of the magnetic charges. The eigenvalues of the translation generators are not naturally quantised, so by interpreting quantum states of magnetic monopoles as eigenstates of the translation operators we select a subset of translation eigenstates by hand, namely those with integer eigenvalues K and half-integer values for the magnitude of $\mathbf{k}$. We return to the group theoretical interpretation of this quantisation in the final section of this paper. Now we impose the quantisation and press on, turning to the computation of the combination rules of dyonic states. These can now be found using the Clebsch-Gordan coefficients of $U(2) \ltimes \mathbf{R}^4$. The interesting part of this is again the (double cover) of the Euclidean group, whose Clebsch-Gordan coefficients can be found in the literature see e.g. [43]. Their calculation is lengthy (remarkably we could not find them in any group theory textbook) because of the subtleties of combining representations with different orbits and centraliser representations. To give a flavour of the subject we note the following Clebsch-Gordan series for the tensor product of two representations. Multiplying two representation with non-vanishing orbit sizes k_1 and k_2 one finds:

$$V_{k_1,s_1} \otimes V_{k_2,s_2} = \int_{|k_1-k_2|}^{k_1+k_2} dk \sum_{t=-\infty}^{\infty} V_{k,s_1+s_2+t}, \tag{10.26}$$

where $k = 0$ is allowed on the right hand side; in that case the sum over $s_1 + s_2 + t$ is a sum over $SU(2)$ representations and should be restricted to positive integers. Physically one may think of this formula in terms of combining plane waves with wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ and helicities s_1 and s_2. If the magnitudes of the wave vectors are fixed to be k_1 and k_2 the combined plane wave may have wave vectors with length k varying between $|k_1 - k_2|$ and $k_1 + k_2$ and a helicity which is integer or half-odd integer depending on the values of s_1 and s_2 but otherwise arbitrary. If $k_1 = k > 0$ and $k_2 = 0$, the tensor product splits into a finite sum of irreducible representations:

$$V_{k,s} \otimes V_{0,j} = \bigoplus_{n=-j}^{j} V_{k,s+n}. \tag{10.27}$$

Finally the case $k_1 = k_2 = 0$ reproduces the familiar $SU(2)$ Clebsch-Gordan series:

$$V_{0,j_1} \otimes V_{0,j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} V_{0,j}. \tag{10.28}$$

We are actually interested in more detailed information, namely the Clebsch-Gordan coefficients which specify the relation between states. On the other hand we only need to consider very special states, namely those singled out by the Dirac quantisation condition on the magnetic charge. This condition should be imposed on both the states to be multiplied and on the resulting product state, and this drastically reduces the number of tensor products we need to consider. Here we have only treated quantum states of monopoles of topological magnetic charge 1 and 2, but we propose as a general requirement that the non-abelian magnetic charges $\mathbf{k}_1$ and $\mathbf{k}_2$ of two states to be multiplied must be parallel or anti-parallel. It is clear that then all states obtained as the tensor of two states which individually satisfy the Dirac condition will also satisfy that condition.

Starting with quantum states of topological charge one monopoles $|1, \mathbf{k}_1; N_1, s_1\rangle$ and $|1, \mathbf{k}_2; N_2, s_2\rangle$ for example, where $k_1 = k_2 = 1/2$ and $N_1 + 2s_1 = N_2 + 2s_2 = 0$, we may thus only combine them if either $\mathbf{k}_1 = \mathbf{k}_2$ or $\mathbf{k}_1 = -\mathbf{k}_2$. In the former case the tensor product is

$$|1, \mathbf{k}; N_1, s_1\rangle \otimes |1, \mathbf{k}; N_2, s_2\rangle = \delta^2(0)|2, 2\mathbf{k}; N_1 + N_2, s_1 + s_2\rangle, \tag{10.29}$$

where the infinite factor $\delta^2(0)$ arises because we are working with non-normalisable states. The equations (10.29) is essentially the combination rule for monopoles with abelian magnetic charges: the direction of $\mathbf{k}$ does not come into play in any interesting way. This agrees with the fact that the quantum states in (10.29) can all be realised as wavefunctions on the smallest stratum of the relevant moduli space and that these strata are fibred of the magnetic orbit, with fibres being isomorphic to $SU(2)$ monopole moduli spaces. The combination rule (10.29) is the combination rule of $SU(2)$ monopoles in a fixed fibre. The possibility of anti-parallel non-abelian magnetic charges is much more interesting. Now the tensor product is

$$|1,\mathbf{k};N_1,s_1\rangle \otimes |1,-\mathbf{k};N_2,s_2\rangle$$
$$= \delta^2(0) \sum_{j=|s_1-s_2|}^{\infty} \sum_{m=-j}^{j} \sqrt{2j+1}\, d^j_{m(s_1-s_2)}(\beta) e^{-im\alpha}\ |2,0;N_1+N_2,j,m\rangle_{s_1-s_2}, \tag{10.30}$$

where (α,β) are again the angles determining the direction of $\mathbf{k}$ as in (3.4). Here the representation theory of $U(2) \ltimes \mathbf{R}^4$ has entered in a non-trivial way and has solved one of the main puzzles of non-abelian dyon physics, namely how to combine two dyons carrying non-abelian magnetic charge and $U(1)\times U(1)$ electric charge into a dyon carrying only topological magnetic charge and a $U(2)$ representation. Particularly we now also see how to decompose a quantum state $|2,0;N,j,m\rangle$ of a charge two monopole in terms of tensor product states of charge one monopoles

$$\delta^2(0)|2,0;N,j,m\rangle_s =$$
$$\int \sin\beta d\beta d\alpha\ \frac{\sqrt{2j+1}}{4\pi} d^{j*}_{ms}(\beta) e^{im\alpha}\ |1,\mathbf{k};N_1,s_1\rangle \otimes |1,-\mathbf{k};N_2,s_2\rangle, \tag{10.31}$$

with the condition $N_1+N_2=N$ and $s_1-s_2=s$. Note that *all* magnetic directions are needed on the right hand side. If we had only considered single monopole states with a particular magnetic directions we would not be able to make sense of general charge two monopole quantum states in terms of tensor product states of charge one monopoles.

We have not discussed monopoles of topological charge three here, but for completeness we also write down the rule for combining a quantum state of a charge one monopole with a quantum state of a charge two monopole in the large stratum. The former carries $U(1)$ electric charge and latter $U(2)$ electric charge, but within the representation theory of $U(2)\ltimes \mathbf{R}^4$ there is no problem in combining such states. The answer is

$$|1,\mathbf{k};N_1,s_1\rangle \otimes |2,0;N_2,j,m\rangle_{s_2} = \frac{\sqrt{2j+1}}{4\pi} e^{im\alpha} d^{j*}_{ms_2}|3,\mathbf{k};N_1+N_2,s_1+s_2\rangle, \tag{10.32}$$

with $\mathbf{k}$ of length 1/2 and direction given by (α,β).

Having demonstrated the use of interpreting dyonic quantum states of $SU(3)$ monopoles as elements of $U(2) \ltimes \mathbf{R}^4$ representations we end this section by highlighting two restrictions which are dictated by the physics of monopoles but which are not naturally part of $U(2) \ltimes \mathbf{R}^4$ representation theory. The first restriction is the superselection rule that states with different (topological or non-abelian) magnetic charge may not be superimposed. The second is a consequence of the Dirac condition and is the restriction on which states may be multiplied in tensor products.

11. BPS quantum states and S-duality

Maximally broken $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is widely believed to enjoy exact invariance under S-duality transformations. In general but precise terms this statement means the following. For a given gauge group G and symmetry breaking to a group H consider the two real parameters which uniquely characterise the action of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, namely the coupling constant e and the θ-angle. Then construct the complex number $\tau = \theta/2\pi + 4\pi i/e^2$ in the upper half plane. An S-duality transformation is a modular transformation on τ

$$\tau \to \frac{q\tau - r}{-p\tau + s}, \tag{11.1}$$

where $M = \begin{pmatrix} q & -r \\ -p & s \end{pmatrix} \in SL(2, \mathbf{Z})$, together with a suitable $SL(2, \mathbf{Z})$ action on the BPS states of the quantum theory (to be defined presently). If H is a maximal torus of G, the BPS states are dyonic states characterised by $R = \text{rank}(H) = \text{rank}(G)$ pairs of integers (m_l, n_l), $l = 1, ..., R$, giving the magnetic and electric charges respectively [4]. Under S-duality these states transform as

$$(m_l, n_l) \to (m_l, n_l) M^{-1} \tag{11.2}$$

for all $l = 1, ..., R$. In particular the electric-magnetic duality operation originally considered by Montonen and Olive [10] is given by the matrix

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{11.3}$$

which exchanges strong with weak coupling and electric with magnetic charges.

Little is known about duality in theories with non-abelian unbroken gauge symmetry. There exists a conjecture, due to Goddard, Nuyts and Olive [6] according to which Yang-Mills-Higgs theory with gauge group G broken to H has a dual description at strong coupling in terms of weakly coupled Yang-Mills-Higgs theory with a dual gauge group $\tilde{G}$ broken to $\tilde{H}$. The GNO conjecture interprets the non-abelian monopole charges, after rotation into the Cartan subalgebra, as labels of irreducible interpretations of the dual group $\tilde{H}$. The true invariance group of Yang-Mills-Higgs theory with unbroken gauge group H would, according to the GNO conjecture, be the product $H \times \tilde{H}$ of an "electric" and a "magnetic" version of the unbroken gauge group. In particular the GNO conjecture would imply that dyonic states fall into representations of that product group. However, here we have seen that dyonic quantum states of monopoles in minimally broken $SU(3)$ Yang-Mills-Higgs theory have correlated magnetic and electric properties, with the magnetic orbit determining the part of the unbroken gauge group with respect to which the dyons carry electric charge. This correlation is not accounted for by the representation theory of the GNO group $U(2) \times \tilde{U}(2)$, but, as we have seen, it is captured perfectly by the representation theory of the semi-direct product $U(2) \ltimes \mathbf{R}^4$. In particular the Clebsch-Gordan coefficients of $U(2) \ltimes \mathbf{R}^4$ lead to combination rules for dyonic states which are consistent with their realisation as wavefunctions on the moduli spaces.

The goal of this section is to look at the implications of our insights into the dyonic spectrum for S-duality in Yang-Mills theory with non-abelian unbroken gauge symmetry. We will consider transformation rules for dyonic states which generalise the abelian rules (11.2) to the non-abelian case with minimal modifications. In particular we thus consider transformation rules which map the dyonic states of minimally broken $SU(3)$ Yang-Mills-Higgs theory into dyonic states of the same theory with dual coupling (11.1). We are aware that this may well be too restrictive. In particular we feel that one needs a better understanding of duality in the massless sector of the $\mathcal{N} = 4$ supersymmetric version of the theory before one can gain a full understanding of duality. However, the massless sector has essentially the same particle contents as unbroken $\mathcal{N} = 4$ supersymmetric $U(2)$ Yang-Mills theory, whose dual formulation is not known [44]. We do not attempt to solve this problem here and focus on the massive particles instead. As we shall see, there is a natural $SL(2, \mathbf{Z})$-duality action on the massive dyonic states, whose study is instructive. We are particularly interested in BPS states, which are (in general) dyonic states in some representation $V_{K,k;N,n}$ of $U(2) \ltimes \mathbf{R}^4$ whose energy equals the BPS bound

$$E_{\mathrm{BPS}} = ve|N + \tau K|. \tag{11.4}$$

where v is the magnitude (2.14) of the vacuum expectation value of the Higgs field.

The starting point for the transformation rule we want to consider is the observation of the previous section that dyonic quantum states fit into certain representations of $U(2) \ltimes \mathbf{R}^4$. This generalises in the obvious way to dyonic states in Yang-Mills-Higgs theory with general unbroken gauge group H: these states fall into certain representations of $H \ltimes \mathbf{R}^D$ (D =dim(H). Thus (as we shall explain in more detail in [9]) they are labelled by R quantised orbit parameters and by R labels specifying the representation of the centraliser group associated to the orbit (which always contains a maximal torus and therefore has the same rank as H). This is also (trivially) true in the maximal symmetry breaking case, where the orbits are always just points in a R-dimensional lattice and the centraliser is always the whole of H. In that case, as we saw above, electric-magnetic duality acts by exchanging orbit labels with centraliser representation labels. In this formulation the electric-magnetic duality transformation rule naturally generalises to the situation where H is non-abelian. Again we focus on the case $G = SU(3)$ and $H = U(2)$.

In that case we have seen that magnetic orbits are labelled by the pair (K, k), with $K \in \mathbf{Z}$ and $0 \leq k \leq K/2$ integer or half-odd integer depending on whether K is even or odd, which may be summarised as the requirement that $K + 2k$ is even. Centraliser representations are labelled by the pair (N, n), where $N \in \mathbf{Z}$ specifies a $U(1)$ representation and n defines a centraliser representation as specified after (10.22). Again we have a $\mathbf{Z}_2$ condition relating N and n, namely that $N + 2n$ is even. The S-duality action we want to consider involves the $SL(2, \mathbf{Z})$ action on the τ-parameter (11.1) together with the following action on the representation labels:

$$\begin{aligned} (K, N) &\to (K, N)M^{-1} \\ (k, n) &\to (k, n)M^{-1}. \end{aligned} \tag{11.5}$$

Recall that in the abelian case, where all representations are one-dimensional, there is either no or a unique BPS state (or, more precisely, a unique short $\mathcal{N} = 4$ supermultiplet) in the representation labelled by (m_l, n_l). Thus there is no difference between an $SL(2, \mathbf{Z})$ action on representations and an $SL(2, \mathbf{Z})$ action on states. In the non-abelian case, however, we have to decide whether we consider the action of S-duality on the entire carrier space of representations or just on certain states in those spaces. A moment's thought shows that one cannot expect the carrier spaces of the representations related by (11.5) to be "physically equivalent": for example the purely electric representation $V_{0,0;1,1/2}$ is two-dimensional whereas the purely magnetic representation $V_{1,1/2;0,0}$ is infinite-dimensional. On may hope that the BPS condition will select the same number of states in all representations related by (11.5), but this is not case as we shall see. We postpone a full discussion of this puzzle until the end of this section. Here we note that the alternative route, that of considering the action of S-duality on individual states in the spaces $V_{K,k;N,n}$, can be implemented quite easily. The idea is to select a unique state in each carrier space $V_{K,k;N,n}$ by a "natural" condition (which necessarily breaks the $U(2) \ltimes \mathbf{R}^4$ invariance of the carrier spaces maximally). One can do this, for example, by picking a unit vector $\hat{\mathbf{k}}$ and in each $V_{K,k;N,n}$ select that state which is the eigenstate of the $SU(2)$ generator $\hat{\mathbf{k}}\cdot\mathbf{I}$ with maximal (or minimal) eigenvalue. If $k = 0$ this condition selects the state in the spin j representation of $SU(2)$ with spin j along the $\hat{\mathbf{k}}$ axis; if $k > 0$ it selects the state $|K, k\hat{\mathbf{k}}; N, n\rangle$ (provided $n \geq 0$). While this condition may appear *ad hoc* we will see at the end of this section that it has played an important role in earlier discussions of electric-magnetic duality.

Consider now the $SL(2,\mathbf{Z})$ orbit of the massive W-bosons of the theory on which previous discussions of electric-magnetic duality have traditionally focused. The massive W-bosons form a doublet under $SU(2)$ and carry one unit of $U(1)$ charge, so they belong to the representation $(K, k) = (0, 0)$ and $(N, n) = (1, 1/2)$. Under the $SL(2,\mathbf{Z})$ transformation (11.5) this is mapped to

$$(K, k) = p(1, \tfrac{1}{2}) \qquad (N, n) = q(1, \tfrac{1}{2}), \tag{11.6}$$

for relatively prime integers p and q. These states carry magnetic charge, and at weak coupling should be visible semi-classically. More precisely, since $k = K/2$, they should emerge as dyonic states in the smallest strata of the moduli spaces we have described. Since these moduli spaces consist of embedded $SU(2)$ monopoles we can hope to deduce the prediction of S-duality in this case from the S-duality properties of the $SU(2)$ theory. This is in fact what we will do, but since our computation needs to be carried out in the $\mathcal{N} = 4$ supersymmetric version of the purely bosonic theory we have described so far, we first need to explain how $\mathcal{N} = 4$ supersymmetry is implemented in our quantisation scheme.

The general procedure for implementing supersymmetry in the collective coordinate or moduli space quantisation of monopole dynamics is explained in [45] in the context of

$SU(2)$ monopoles. For $\mathcal{N}=4$ supersymmetry this procedure leads to the Hilbert space of states being the space of all square-integrable real differential forms on the moduli space and the Hamiltonian being the covariant Laplacian. An important consistency requirement is that the metric on the moduli space is hyperkähler. The arguments given in [45] apply without essential changes to the largest strata of the moduli spaces we have described, i.e. those labelled by K even and $k=0$. These spaces have dimension $6K$ (which is thus divisible by four) and are equipped with smooth hyperkähler metrics. All the other strata, however, have dimensions which are even but not divisible by four, indicating clearly that the standard procedure needs to be amended. In fact the required changes follow naturally from the magnetic superselection rules introduced in the previous section. Namely, the Hilbert spaces $\mathcal{H}_{K,\mathbf{k}}$ introduced after (10.4) can be extended to accommodate $\mathcal{N}=4$ supersymmetry in the standard fashion. Their elements are functions on the fibres $M_{K,\mathbf{k}}$ of the fibration (4.9) and these spaces do have dimensions which are multiples of four. In the case $k=K/2$ they are actually isomorphic to $SU(2)$ monopole moduli spaces and thus in particular equipped with hyperkähler metrics. It has not been shown rigorously that for $k<K/2$ these spaces also have smooth hyperkähler metrics, but non-rigorous physical arguments suggest that they do. Here we are in any case mainly interested in the case $k=K/2$, so we fix this condition from now on.

Then we define the space $\mathcal{H}^*_{K,\mathbf{k}}$ of square-integrable real differential forms on $M_{K,\mathbf{k}}$. The quantum Hamiltonian is the Laplacian on $M_{K,\mathbf{k}}$, which equals, up to scaling, the Laplacian on the $SU(2)$ monopole moduli space $M_K^{\mathrm{SU(2)}}$. The eigenstates of that Hamiltonian which saturate the BPS bound, however, were conjectured by Sen [2] to exist, and be unique, for all $(K,N)=(p,q)$ relatively prime integers. There is now much support for the validity of the Sen conjecture [2], [3], and assuming it we deduce the existence of a unique dyonic BPS state for each value of the non-abelian magnetic charge $\mathbf{k}$ in the $U(2)\ltimes\mathbf{R}^4$ representation $V_{K,K/2;N,N/2}$ with $(K,N)=(p,q)$ relatively prime integers. Thus we indeed find the BPS states on the $SL(2,\mathbf{Z})$ orbit of the massive W-boson states, but we encounter the puzzle anticipated earlier in this section: whereas the massive W-bosons fill the two-dimensional representation $V_{0,0;1,1/2}$, the dyonic BPS states fill the infinite-dimensional representations $V_{K,K/2;N,N/2}$.

In earlier discussion of duality e.g. in [6] this "dimensionality puzzle" is not seen because only vector magnetic charges on the orbit of a given vector magnetic charge under the action of the Weyl group of the unbroken gauge group are considered. It follows from general group theory that the number of points on the Weyl orbit of the vector magnetic charge of a topological charge one monopole is equal to the dimension of the fundamental representation of the unbroken gauge group. Thus, with this counting procedure the number of massive W-boson states agrees with the number of topological charge one monopoles. This observation is in fact a key ingredient in the formulation of the GNO conjecture in [6]. Here, however, we have seen that the full magnetic orbit is essential for understanding dyonic states and their interaction, and that any restriction to a subset of the orbit is artificial and will miss some of the physics of dyons. From our point of view the restriction to Weyl orbits is equivalent to choosing a direction $\hat{\mathbf{k}}$ and restricting attention to states which are eigenstates of the operator $\hat{\mathbf{k}}\cdot\mathbf{I}$ introduced earlier in this section. Thus GNO duality in the theory considered here would, in our language, amount to choosing $\hat{\mathbf{k}} = (0, 0, 1)$ and selecting the eigenstates of I_3 with maximal and minimal eigenvalues. This condition selects the usual basis of massive W-boson states in the purely electric sector $V_{0,0;1,1/2}$ while in the magnetic sector $V_{1,1/2;0,0}$ it picks out two monopoles with magnetic charges on the same Weyl orbit (the Weyl group of $U(2)$ is the permutation group S_2 of two elements and acts on the magnetic orbits we have defined by sending a point to its antipodal point). Such a choice can be made, but it is *ad hoc* and breaks the symmetry of the theory in an artificial way.

We end this section with two comments. The first concerns the place of Weyl orbits in our description of dyons. Although we do not select any *particular* Weyl orbit in our scheme, Weyl orbits do play an important role. Dyonic states with non-abelian magnetic charges in the same Weyl orbit have parallel or anti-parallel non-abelian magnetic charges and are thus precisely those states in $V_{K,k;N,s}$ which may be multiplied in a tensor product. Thus, although the representation space $V_{K,k;N,s}$ is infinite-dimensional for $k > 0$, a given state can interact consistently with at most two other states in the same representation. This is intriguingly reminiscent of the role Weyl orbits play in the physics of classical non-abelian electric charges. In [46] points on Weyl orbits were shown to correspond one-to-one to the relative orientations which two non-abelian electric point charges may have if the non-abelian electric field they produce is to be static. These observations are not a sufficient reason to assign any dimension other than infinity to the carrier spaces $V_{K,k;N,s}$ $(k > 0)$ but they do show that counting degrees of freedom for particles carrying non-abelian charges is a subtle business.

The second comment concerns the paper [11]. In that paper, a regularisation procedure was used to make motion on (in our language) the magnetic orbit dynamically possible and a supersymmetric quantisation scheme was adopted which allowed forms on the magnetic orbit as physical states. In that scheme computing the degeneracies of magnetic states is equivalent to counting harmonic forms on the magnetic orbit, i.e. to computing its Euler characteristic. However, the Euler characteristic of the coset spaces considered in [11] is equal to the number of points in the orbit under the Weyl group of the unbroken gauge group of any point in the coset. We relegate a discussion and proof (which is a simple application of Morse theory) of this statement for general gauge groups to [9], but in the theory considered in this paper the validity is easily demonstrated. Here the relevant coset is the magnetic orbit whose Euler characteristic is two if $k > 0$ and one if $k = 0$. Similarly the orbits of the Weyl group acting on the magnetic orbit have order two if $k > 0$ and order one if $k = 0$. Thus, the counting procedure for magnetic states described in [11] also boils down to determining the order of Weyl orbits.

12. Discussion and outlook

The mathematical structure of the monopole moduli spaces described in this paper singles out certain coordinates, called magnetic orbits here, and only allows smooth isometric actions of a subgroup of the unbroken gauge group, namely the centraliser group associated to the orbit. As a result magnetic orbits and electric centraliser representations turn out to be the natural labels of dyonic quantum states in minimally broken $SU(3)$ Yang-Mills-Higgs theory. Noting that orbits and centraliser representations are also the attributes of representations of the semi-direct product $U(2) \ltimes \mathbf{R}^4$ we have proposed to interpret dyonic states as elements of such representations. It is worth stressing again that this group is not a symmetry of the classical theory and that in particular the magnetic part $\mathbf{R}^4$ does not act on either the classical configuration space or the moduli spaces. Nonetheless we have seen that the wavefunctions on the moduli space are acted upon by the full group $U(2) \ltimes \mathbf{R}^4$. Thus, although one usually thinks of the electric excitations as quantum effects and the magnetic properties as classical, we find that in our scheme the magnetic properties are also encoded in quantum wavefunctions. Furthermore, the interpretation of dyonic states as elements of representations of $U(2) \ltimes \mathbf{R}^4$ leads to combination rules of dyonic states which are consistent with the realisation of dyonic states as wavefunctions on the appropriate moduli spaces.

However, while thinking of dyonic states as elements of $U(2) \ltimes \mathbf{R}^4$ representations turns out to be very fruitful, it also raises a number of questions, the most important of of which concerns the quantisation of the magnetic orbit sizes: from the point of view of $U(2) \ltimes \mathbf{R}^4$ representations this is an artificial condition, while in monopole physics it is one of the most basic facts. Ideally we would like the algebraic object which classifies dyonic quantum states to incorporate quantised magnetic orbit sizes (or equivalently the Dirac quantisation condition) as a basic ingredient. It remains a challenge to find that algebraic object, which we expect to be closely related to semi-direct product groups.

A second question concerns the S-duality properties of the dyonic spectrum. We have seen that it is possible to define an $SL(2, \mathbf{Z})$-action on the dyonic states of the theory, with the electric-magnetic duality operation exchanging labels of the magnetic orbits with labels of the centraliser representations. With this definition the basic massive W-boson states are seen to be part of an $SL(2, \mathbf{Z})$-orbit whose other elements are dyonic BPS states which can be found semi-classically as quantum states on the smallest strata of monopole moduli spaces. However, in interpreting these BPS states as elements of $U(2) \ltimes \mathbf{R}^4$ representations we are forced to conclude that S-duality relates states belonging to $U(2) \ltimes \mathbf{R}^4$ representations of different dimensions. We expect that a solution of this "dimensionality puzzle" will require a deeper understanding of S-duality in supersymmetric Yang-Mills theory with non-abelian unbroken gauge symmetry.

To end, we want to stress a link to physics in (2+1) dimensions. In [47] it was shown that dyonic quantum states of vortices in (2+1)-dimensional gauge theories with discrete unbroken gauge groups are labelled by magnetic fluxes together with a centraliser representation of the unbroken gauge group. There the relevant algebraic object whose representations classify the dyonic states is the so-called quantum double $D(H)$ of the unbroken gauge group H, which by definition is the tensor product of the algebra of functions on the group with the group algebra. Dyons in two spatial dimensions have distance-independent topological interactions (Aharanov-Bohm scattering, flux metamorphosis) and remarkably these can be deduced from the algebraic structure of $D(H)$. More recently quantum doubles have also been constructed for continuous groups, such as $SU(2)$ [48] and were found to have a number of structural similarities to semi-direct product groups of the type discussed here, with the function algebra in the quantum double being related to the translation part of the semi-direct product groups. The intriguing relation between these two algebraic objects deserves to be studied further. The important lesson one learns from studying quantum states of both vortices with non-abelian flux and monopoles with non-abelian magnetic charge is that consistent algebraic classifications of these states require that one treats magnetic and electric properties as interdependent aspects of one algebraic object.

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