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Skyrme-Maxwell Solitons in (2+1) Dimensions

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Abstract

A gauged (2+1)-dimensional version of the Skyrme model is investigated. The gauge group is $U(1)$ and the dynamics of the associated gauge potential is governed by a Maxwell term. In this model there are topologically stable soliton solutions carrying magnetic flux which is not topologically quantized. The properties of rotationally symmetric solitons of degree one and two are discussed in detail. It is shown that the electric field for such solutions is necessarily zero. The solitons' shape, mass and magnetic flux depend on the $U(1)$ coupling constant, and this dependence is studied numerically from very weak to very strong coupling.

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I Introduction

The Skyrme model is a generalized non-linear sigma model in (3+1) dimensions [1]. It has soliton solutions which, after suitable quantization, are models for physical nucleons [2]. The theory is invariant under the group $SO(3)_{\text{iso}}$ of iso-rotations, and electromagnetism is introduced into the model by gauging a $U(1)$ subgroup of $SO(3)_{\text{iso}}$, see [3] for details. The resulting fully coupled Skyrme-Maxwell system is mathematically hard to analyze, but of considerable physical interest: it is here that one should compute the Skyrme model's prediction for the proton-neutron mass difference for example. In fact such a computation (which necessarily involves quantum theory) was first attempted in [4] where the authors made various approximations based on the smallness of the fine structure constant. The result was a mass difference of $m_p - m_n = 1.08$ MeV, which has the wrong sign (in nature the proton is 1.29 MeV lighter than the neutron).

In this paper we investigate classical properties of a (2+1)-dimensional version of the gauged Skyrme model. The model, to be introduced in section II, is a gauged version of the baby Skyrme model studied in [5] and [6] and contains a dynamical abelian gauge field. It has soliton solutions which are stable for topological reasons and which carry magnetic flux. However, the gauge symmetry is unbroken and the solitons differ from the much studied flux tubes or vortices in the abelian Higgs model in that their magnetic flux is not quantized. Contrary to the situation in (3+1)-dimensional Skyrme-Maxwell theory it is possible to compute certain soliton solutions in our model explicitly with moderate numerical effort, and to investigate their structure quantitatively. Thus we study the dependence of the magnetic flux and the solitons' mass on the electromagnetic coupling constant, assess the back-reaction of the electromagnetic field on the matter fields and make some semi-quantitative statements about the long range inter-soliton forces.

II Gauging the Baby Skyrme Model

The model we want to study is defined on (2+1)-dimensional Minkowski space, whose signature we take to be $(-,+,+)$. Points in Minkowski space are written as $(t, \mathbf{x})$ or simply x, with coordinates x^α, $\alpha = 0, 1, 2$, and the velocity of light is set to 1. We will mostly be concerned with static fields and sometimes use the label $i = 1, 2$ for the coordinates of the spatial vector $\mathbf{x}$. The basic fields are a scalar field $\boldsymbol{\phi}$ describing matter and a $U(1)$ gauge potential A_α for the electromagnetic field. More precisely, $\boldsymbol{\phi}(x)$ is a 3-component vector satisfying the constraint $\boldsymbol{\phi} \cdot \boldsymbol{\phi} = \phi_1^2 + \phi_2^2 + \phi_3^2 = 1$, thus lying on a 2-sphere which we denote by $\mathcal{S}^2_\phi$.

The model is a gauged version of the baby Skyrme model studied in detail in [5] and [6]. The Lagrangian considered there is invariant under global iso-rotations of the field $\boldsymbol{\phi}$ about a fixed axis $\mathbf{n} \in \mathcal{S}^2_\phi$. Taking $\mathbf{n} = (0, 0, 1)$ for definiteness such a rotation can be written in terms of the rotation angle $\chi \in [0, 2\pi)$ as

$$(\phi_1, \phi_2, \phi_3) \to (\cos\chi\, \phi_1 + \sin\chi\, \phi_2, -\sin\chi\, \phi_1 + \cos\chi\, \phi_2, \phi_3)\,. \tag{1}$$

We write $SO(2)_{\text{iso}}$ for the group of such rotations.

Here we couple electromagnetism to the baby Skyrme model by gauging the $SO(2)_{\text{iso}}$ symmetry. Thus we require invariance under local rotations

$$\boldsymbol{\phi} \to O(x)\boldsymbol{\phi}\,, \tag{2}$$

where $O(x)$ is an $SO(2)_{\text{iso}}$ rotation matrix which depends on x. For infinitesimal rotation angles $\varepsilon(x)$, this becomes

$$\boldsymbol{\phi} \to \boldsymbol{\phi} + \varepsilon \mathbf{n} \times \boldsymbol{\phi}. \tag{3}$$

The abelian gauge field A_α transforms as $A_\alpha \to A_\alpha - \partial_\alpha \varepsilon$, so we define the covariant derivative via

$$D_\alpha \boldsymbol{\phi} = \partial_\alpha \boldsymbol{\phi} + A_\alpha \mathbf{n} \times \boldsymbol{\phi}. \tag{4}$$

Finally we define the curvature or field strength $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ with electric components $E_i = F_{i0}$ and the magnetic component $B = F_{12}$. Thus we can write down the Lagrangian of our model:

$$L = -H \int d^2x \left(\frac{1}{2}(D_\alpha \boldsymbol{\phi})^2 + \frac{\lambda^2}{4} \left(D_\alpha \boldsymbol{\phi} \times D_\beta \boldsymbol{\phi}\right)^2 + \mu^2 \left(1 - \mathbf{n} \cdot \boldsymbol{\phi}\right) + \frac{1}{4g^2} F_{\alpha\beta}^2 \right). \tag{5}$$

The first term is a gauged version of the $O(3)$ sigma model Lagrangian (see [7]), the second is a gauged Skyrme term, the third term may physically be thought of as a pion mass term [6] and the last term is the usual Maxwell Lagrangian. There are 4 free parameters in this model. H has the dimension energy, λ and $1/\mu$ are of dimension length while g represents the coupling strength to the gauge field and is also of dimension length. We will discuss our choice of parameters in further detail below, but for the time being we fix our energy scale by setting $H = 1$.

It is worth recalling that the Skyrme term is necessary in the (ungauged) Skyrme model to prevent soliton solutions from collapsing to singular spikes. In the gauged model, however, the Maxwell term has the same scaling behaviour as the Skyrme term, which suggests that there could be stable solitons in a "Skyrme-Maxwell theory without a Skyrme term". This possibility is studied in [8]. Here we retain the Skyrme term because we are also interested in properties of soliton solutions in the limit of vanishing electromagnetic coupling. The Skyrme term ensures the existence of stable solitons in this limit.

The Euler-Lagrange equations for this model can be written conveniently in terms of

$$\mathbf{J}_\alpha = \boldsymbol{\phi} \times D_\alpha \boldsymbol{\phi} + \lambda^2 D_\beta \boldsymbol{\phi} (D^\beta \boldsymbol{\phi} \cdot \boldsymbol{\phi} \times D_\alpha \boldsymbol{\phi}) \tag{6}$$

and the conserved current

$$j_\alpha = \mathbf{n} \cdot \mathbf{J}_\alpha \,. \tag{7}$$

They read:

$$\begin{aligned} D_\alpha \mathbf{J}^\alpha &= \mu^2 \mathbf{n} \times \boldsymbol{\phi} && (8) \\ \partial_\alpha F^{\alpha\beta} &= g^2 j^\beta \,. && (9) \end{aligned}$$

The second equation has three components which we want to write explicitly in terms of the electric and magnetic field, for later use. The $\beta = 0$ component is Gauss's law:

$$\partial_i E_i = j_0 . \tag{10}$$

The remaining two equations are particularly simple when expressed in terms of polar coordinates (r, θ) for $\mathbf{x}$. Defining polar and radial coordinates of the current j via $j_\theta = x_1 j_2 - x_2 j_1$, $j_r = (x_1 j_1 + x_2 j_2)/r$ and analogously for the electric field E_i, we obtain

$$\frac{\partial E_r}{\partial t} - \frac{1}{r}\frac{\partial B}{\partial \theta} \;=\; g^2 j_r$$

$$\frac{\partial E_\theta}{\partial t} + r\frac{\partial B}{\partial r} \quad = \quad g^2 j_\theta \,. \tag{11}$$

The energy E of a configuration $(\boldsymbol{\phi}, A_\alpha)$ is the sum of the kinetic energy

$$T = \frac{1}{2}\int d^2x \left((D_0\boldsymbol{\phi})^2 + \frac{\lambda^2}{2}(D_0\boldsymbol{\phi} \times D_i\boldsymbol{\phi})^2 + \frac{1}{g^2}E_i^2 \right) \tag{12}$$

and the potential energy

$$V = \frac{1}{2}\int d^2x \left((D_1\boldsymbol{\phi})^2 + (D_2\boldsymbol{\phi})^2 + \lambda^2(D_1\boldsymbol{\phi} \times D_2\boldsymbol{\phi})^2 + 2\mu^2(1 - \mathbf{n}\cdot\boldsymbol{\phi}) + \frac{1}{g^2}B^2 \right). \tag{13}$$

In this paper we are only interested in finite energy configurations, so we require that for all t

$$\lim_{r\to\infty} \boldsymbol{\phi}(t, \mathbf{x}) = \mathbf{n}\,. \tag{14}$$

This boundary condition allows the Euclidean space $\mathbf{R}^2$ to be compactified to a topological 2-sphere $\mathcal{S}^2_{\mathrm{x}}$ so that, at a given time t, fields $\boldsymbol{\phi}$ may topologically be regarded as maps

$$\boldsymbol{\phi} \quad : \quad \mathcal{S}^2_{\mathrm{x}} \to \mathcal{S}^2_{\phi}\,. \tag{15}$$

Such maps are topologically classified by their degree Q, which is an integer and can be calculated via

$$Q = \frac{1}{4\pi}\int_{\mathcal{S}^2_{\mathrm{x}}} d^2x\, \boldsymbol{\phi}\cdot(\partial_1\boldsymbol{\phi} \times \partial_2\boldsymbol{\phi})\;. \tag{16}$$

The degree is a homotopy invariant of $\boldsymbol{\phi}$ and therefore conserved during time evolution.

III Static Solutions

It is a well-known and important feature of the $O(3)$ sigma model and its generalization to baby Skyrme models that the potential energy of a configuration is bounded below by the modulus of its degree (or a suitable multiple thereof). A similar result holds in our model, but its proof requires a little work. In [8], where the the potential energy functional

$$V_{\mathrm{aux}}[\boldsymbol{\phi}, A_i] = \frac{1}{2}\int d^2x\, \left((D_1\boldsymbol{\phi})^2 + (D_2\boldsymbol{\phi})^2 + (1 - \mathbf{n}\cdot\boldsymbol{\phi})^2 + B^2 \right) \tag{17}$$

is studied in detail, it is shown that $V_{\mathrm{aux}}[\boldsymbol{\phi}, A_i]$ is bounded below by $4\pi|Q|$. Thus, changing variables in the expression for V via $\mathbf{x} \to \mu\mathbf{x}$, and discarding the positive definite Skyrme term we have the inequality

$$V[\boldsymbol{\phi}, A_i] \geq \frac{1}{2}\int d^2x \left((D_1\boldsymbol{\phi})^2 + (D_2\boldsymbol{\phi})^2 + 2(1 - \mathbf{n}\cdot\boldsymbol{\phi}) + \frac{\mu^2}{g^2}B^2\right). \tag{18}$$

Now note that, since $0 \leq (1 - \mathbf{n}\cdot\boldsymbol{\phi}) \leq 2$, it follows that $(1 - \mathbf{n}\cdot\boldsymbol{\phi}) \geq \frac{1}{2}(1 - \mathbf{n}\cdot\boldsymbol{\phi})^2$. Thus we also deduce

$$V[\boldsymbol{\phi}, A_i] \geq \frac{1}{2}\int d^2x \left((D_1\boldsymbol{\phi})^2 + (D_2\boldsymbol{\phi})^2 + (1 - \mathbf{n}\cdot\boldsymbol{\phi})^2 + \frac{\mu^2}{g^2}B^2\right). \tag{19}$$

If $\mu/g \geq 1$ it then follows immediately that

$$V[\boldsymbol{\phi}, A_i] \geq V_{\mathrm{aux}}[\boldsymbol{\phi}, A_i] \geq 4\pi|Q|. \tag{20}$$

If $\mu/g \leq 1$ on the other hand we have

$$V[\boldsymbol{\phi}, A_i] \geq \frac{\mu^2}{g^2}V_{\mathrm{aux}}[\boldsymbol{\phi}, A_i] \geq 4\pi\frac{\mu^2}{g^2}|Q|. \tag{21}$$

In both cases we have therefore found a topological lower bound for the potential energy V.

Our next goal is to find static configurations of given degree $n > 0$ which minimize the potential energy V. To find these we exploit the symmetries of our model. Both V and Q are invariant under spatial rotations and translations $\mathbf{x} \to R\mathbf{x} + \mathbf{d}$, where R is an $SO(2)$-matrix and $\mathbf{d}$ a translation vector in $\mathbf{R}^2$, and under global $SO(2)_{\mathrm{iso}}$ rotations defined earlier (1). They are also invariant under simultaneous reflections in the Euclidean plane and on the $\mathcal{S}^2_\phi$ manifold:

$$\begin{aligned} (x_1, x_2) &\to (-x_1, x_2) \\ (\phi_1, \phi_2, \phi_3) &\to (-\phi_1, \phi_2, \phi_3)\ . \end{aligned} \tag{22}$$

Physically one may think of this transformation as simultaneous electric charge conjugation and parity operations.

Translationally invariant fields necessarily have degree zero, but it is possible to write down fields of arbitrary degree which are invariant under the reflection (22) and under a combination of a rotation by some angle $\chi \in [0, 2\pi)$ and an iso-rotation by $-n\chi$. The appropriate ansatz for the scalar field $\boldsymbol{\phi}$ is, in terms of polar coordinates (r, θ) for $\mathbf{x}$,

$$\boldsymbol{\phi}(r,\theta) = \begin{pmatrix} \sin f(r) & \cos n\theta \\ \sin f(r) & \sin n\theta \\ \cos f(r) & \end{pmatrix} . \tag{23}$$

This is a two-dimensional version of the hedgehog ansatz used in the three dimensional Skyrme model. Here we consider configurations with their symmetry center, defined by $\boldsymbol{\phi} = -\mathbf{n}$, at the origin.

Under the reflection (22) the gauge field transforms as

$$A_0 \mapsto -A_0, \qquad A_r \mapsto -A_r, \qquad A_\theta \mapsto A_\theta, \tag{24}$$

with the polar and radial coordinates of A_i defined analogously to those of j_i before Eq. (11). Thus the requirement of rotational symmetry and reflection symmetry implies the following form for the gauge potential

$$A_0 = A_r = 0\,, \qquad A_\theta = na(r)\,, \tag{25}$$

where a is an arbitrary function and the factor n is introduced for convenience. For such a gauge field the electric field vanishes, and the magnetic field is given by

$$B = n\frac{a'}{r}. \tag{26}$$

To ensure that the field is regular at the origin we impose

$$f(0) = k\pi\,, \quad k \in \mathbb{Z} \quad \text{and} \quad a(0) = 0\,, \tag{27}$$

and the finite energy requirement implies for the function f

$$\lim_{r\to\infty} f(r) = 0. \tag{28}$$

With these boundary conditions the topological charge Q of the hedgehog configuration (23) is equal to $-n$ if k is odd and zero otherwise [5]. In the following we will restrict attention to $k = 1$.

For configurations of the form (23) and (25) the current j_α has only one non-vanishing component, namely j_θ:

$$j_\theta = n(1+a)(1+\lambda^2 f'^2)\sin^2 f. \tag{29}$$

The electric field vanishes and the magnetic field is independent of θ, so only the θ-component of the Euler-Lagrange Eq. (9) is non-trivial. Thus the field equations (8) and (9) imply two equations for a and f, which read as follows:

$$f''\left(1+\lambda^2\tilde{a}^2\sin^2 f\right) + \frac{f'}{r}\left(\left(2(\tilde{a}r)' - \tilde{a}\right)\lambda^2\tilde{a}\sin^2 f + \lambda^2 r f'\tilde{a}^2\sin f\cos f + 1\right)$$
$$-\tilde{a}^2\sin f\cos f - \mu^2\sin f = 0\,, \tag{30}$$

where $\tilde{a} = n(a+1)/r$, and

$$a'' - \frac{1}{r}a' = g^2(1+a)(1+\lambda^2 f'^2)\sin^2 f\,. \tag{31}$$

Note that $a = 0$ is not a solution of the second equation. Since other constant solutions are forbidden by the boundary condition (27) it follows that all solutions will have a non-trivial magnetic field given by (26).

We will discuss the solutions of (30) and (31) in detail in the next sections, but first we want to address another question of principal interest. Are there finite-energy solutions of the field equations (8) and (9) which have a time-independent purely radial electric field? First we note that, in two spatial dimensions, finite energy solutions necessarily have zero electric charge

$$q = \int d^2x\, j_0(\mathbf{x}). \tag{32}$$

For it follows from Gauss's law (10) that the modulus of the electric field falls off like q/r for large r. Hence the electric energy $\int d^2x\, E_i^2$ diverges if $q \neq 0$. However, this argument does not rule out finite energy solutions with a non-trivial but spherically symmetric charge distribution which integrates to zero. Like for example the hydrogen atom such a distributions would only produce a short range electric field. We claim that this possibility is not realized in our model. To prove this assertion we must allow for more general fields than considered so far. In particular we can no longer impose the reflection symmetry (22) since it eliminates radial electric fields from the start.

Imposing invariance under simultaneous spatial rotations and iso-rotations leads to the following general form for the scalar field:

$$\boldsymbol{\phi}(r,\theta,t) = \begin{pmatrix} \sin f(r) & \cos(n\theta - \chi(r,t)) \\ \sin f(r) & \sin(n\theta - \chi(r,t)) \\ \cos f(r) & \end{pmatrix}, \tag{33}$$

where $\chi(r,t)$ is an arbitrary function of r and t. However, having introduced this function we can immediately remove it by a gauge transformation which brings (33) into the standard hedgehog form (23). Thus having fixed the gauge we write down the most general gauge field which gives rise to a purely radial time-independent electric field

$$A_0 = v(r), \qquad A_r = h(r)\,t, \qquad A_\theta = na(r), \tag{34}$$

where v and h are arbitrary functions of r. The electric field is then

$$E_i = -(v'(r) + h(r))\frac{x_i}{r}. \tag{35}$$

Inserting this ansatz into the field equation (8) and (9) leads to a complicated set of coupled differential equations. Let us first consider the "electromagnetic" equations (9). The θ-component of the current j_α is still given by (29), so the equation implied by the θ-component of (9) is (31) as before. However, both the t and r component of (9) now lead to non-trivial equations which read

$$\begin{aligned} t\,h\,\sin^2 f &= 0 & (36) \\ v'' + \frac{1}{r}v' &= g^2 v(1 + f'^2)\sin^2 f\,. & (37) \end{aligned}$$

The first clearly implies that h is identically zero. To analyze the second we first note that v has to satisfy the boundary condition $v'(0) = 0$ to ensure that the electric field is regular at the origin and that for large r, $v'(r)$ has to tend to 0 faster than $1/r$ for the electric field energy to be finite. However, under these conditions we can multiply (37) by $(r\,v(r))$, integrate both sides over r from 0 to ∞ and finally integrate by parts to obtain

$$\int_0^\infty rdr\,\left((v')^2 + v^2 g^2(1 + \lambda^2 f'^2)\sin^2 f)\right) = 0\,. \tag{38}$$

Since the integrand is positive definite it follows that $v = 0$ everywhere. Thus $v = h = 0$, and the functions f and a satisfy the same equations as before. In particular, the electric field of the solution vanishes everywhere.

IV Asymptotic Properties

To learn more about the minimal energy solutions of the rotation and reflection symmetric form (23) and (25) we need to solve of the boundary value problem posed by the coupled second order equations (30) and (31) and the boundary conditions (27) and (28). This requires a careful analysis of the equations near the regular singular points $r = 0$ and $r = \infty$ of (30) and (31).

At the origin, f and a behave as follows

$$\begin{aligned} f(r) &\approx \pi + c_0 r^n + c_2 r^{n+2} && (39)\\ a(r) &\approx d_0 r^2 + g^2 \frac{c_0^2\,(1+\delta_n^1 c_0^2)}{4n(n+1)} r^{2n+2}\,, && (40)\end{aligned}$$

where δ_n^1 is the Kronecker symbol, c_0 and d_0 are arbitrary parameters and c_2 is a function of c_0, d_0, n and μ.

We already know that f tends to zero for large r (28). Thus the Eq. (31) becomes, for large r,

$$ra'' = a'\,. \tag{41}$$

which is solved by a constant function or by $a(r) = r^2$. Since the latter leads to a magnetic field with infinite energy we conclude that there exist a number a_∞ such that

$$\lim_{r\to\infty} a(r) = a_\infty. \tag{42}$$

Note that the finite energy requirement does not impose any restrictions on the value of a_∞. Since a_∞ is related to the magnetic flux

$$\Phi = \int d^2x\, B \tag{43}$$

via

$$\Phi = 2\pi n a_\infty, \tag{44}$$

there are also no *a priori* restrictions on the value of the magnetic flux. This should be contrasted with the situation in the abelian Higgs model, for example, where the flux is quantized.

It follows from (42) that the Eq. (30) can be simplified for large r, and becomes

$$f'' + \frac{f'}{r} - \left(\frac{n^2(a_\infty + 1)^2}{r^2} + \mu^2\right) f = 0\,. \tag{45}$$

The solutions of this equation are the modified Bessel functions $K_m(\mu r), m = n(a_\infty + 1)$. Thus f is asymptotically proportional to

$$K_m(\mu r) \sim \sqrt{\frac{\pi}{2\mu r}} e^{-\mu r} \left(1 + \mathcal{O}\left(\frac{1}{\mu r}\right)\right)\,. \tag{46}$$

We then deduce from (31) the asymptotic proportionality

$$B(r) \propto -\frac{1}{r} e^{-2\mu r} \left(1 - \mathcal{O}\left(\frac{1}{2\mu r}\right)\right)\,. \tag{47}$$

This shows in particular that the magnetic field has no long range component.

Having understood the asymptotic properties of Eqs. (30) and (31) it is relatively straightforward to solve them numerically. We have done this for $n = 1$ and $n = 2$ using both a shooting method and a relaxation technique, with identical results. The more general study of static multisoliton solutions in the (ungauged) baby Skyrme model in [5] suggests that minimal energy configurations have the rotationally symmetric form considered here for degrees 1 and 2, but are less symmetric for higher degrees. We expect that the rotationally symmetric solutions of degree $n > 2$ are similarly not true minima of the potential energy in our gauged model, and we therefore do not consider them here.

The solution for $n = 1$ corresponds to the basic soliton of our model, which we call a gauged baby Skyrmion. Already the simple asymptotic analysis carried out in this section tells us some of its qualitative features. The profile function f of the gauged baby Skyrmion has the same asymptotic behaviour as that of the the ungauged baby Skyrmion discussed in [5]. As explained there the resulting asymptotic forms of the fields ϕ_1 and ϕ_2 are the same as those produced by two orthogonal *scalar* dipoles in classical Klein-Gordon theory. Thus, from the point of view of the scalar (or matter) fields, the gauged baby Skyrmion looks from afar like a pair of orthogonal scalar dipoles in the plane of motion.

In addition, however, the gauged baby Skyrmion has a non-trivial electric current distribution with a magnetic dipole moment orthogonal to the plane of motion. Such a magnetic dipole moment does not produce a long range magnetic field in (2+1)-dimensional electromagnetism, but it does carry magnetic flux. Our soliton similarly has no long-range magnetic field and also carries magnetic flux. Thus from afar a gauged baby Skyrmion looks like a triplet of mutually orthogonal dipoles: two scalar dipoles in the plane of motion and one magnetic dipole orthogonal to it.

V Numerical Results

To compute explicit solutions of (30) and (31) we need to fix the parameters of the model. We can fix the energy and length scales by setting $H = 1$ and $\lambda = 1$, so that we are working in geometric units where all quantities are dimensionless. We further choose $\mu^2 = 0.1$ in order to be able to compare our results with the discussion of the ungauged baby Skyrme model in [5] (where the reader can read about the motivation for this choice). The electromagnetic coupling constant g, however, is not kept fixed. In fact we are particularly interested in the dependence of the baby Skyrmion's properties on g. Consider first two global properties: the energy or mass, and the magnetic flux Φ (43).

The energy of the solutions for $n = 1$ and $n = 2$ is shown in Fig. 1 as a function of g. When $n = 2$, the energy is less than twice the energy of the $n = 1$ soliton for all g, so this solution may be thought of as a bound state of two gauged baby Skyrmions. Recall that the Bogomol'nyi bound on the energy depends on whether $g \leq \mu$ or $g \geq \mu$. In the first regime the bound is independent of g, but in the second it decreases like $1/g^2$, compare (20) and (21). From Fig. 1 it is clear that the energy shows a very similar dependence on g. For both $n = 1$ and $n = 2$ it is essentially constant in the first regime, staying approximately 50 % above the Bogomol'nyi bound. Here we find in particular that in the limit $g \to 0$ the energy of the $n = 1$ and $n = 2$ solution tends to $E_1 = 1.564 \cdot 4\pi$ and $E_2 = 2.936 \cdot 4\pi$ respectively, which agrees with the calculation for the ungauged model in [5]. In the second regime, by contrast, the energy, like the Bogomol'nyi bound, decreases rapidly as g is increased further. However, our numerical results suggest that for both $n = 1$ and $n = 2$ the energy tends to a non-zero limit for $g \to \infty$.

The precise dependence of the magnetic flux Φ on the coupling constant g is shown in Fig. 2 for both $n = 1$ and $n = 2$. In the limit $g \to 0$ the magnetic flux tends to zero, which is what one expects physically and which one can understand analytically by noting that in the limit $g \to 0$ Eq. (31) becomes Eq. (41). As we saw in our earlier discussion of that equation, the only finite energy solution is the constant solution. It then follows from the boundary condition (27) that a and hence also Φ vanish in this limit. Furthermore, integrating Eq. (31) once and using $\lim_{r\to\infty} B(r) = 0$ we find

$$B(0) = g^2 \int_0^\infty dr \left(\frac{1+a}{r}\right)(1 + \lambda^2 f'^2)\sin^2 f. \tag{48}$$

Numerically we observe that the dependence of a and f on g is small for weak coupling, and that the flux Φ is approximately proportional to $B(0)$, so we expect Φ to grow quadratically with g for small g. This is precisely what the double logarithmic plot in Fig. 2 shows: there is a weak coupling range ($0 \leq g \leq \mu$) where $\log\Phi$ is a linear function of $\log g$ with gradient 2. Thus in this region the magnetic flux of the solutions of degree $n = 1$ and $n = 2$ is approximately

$$\Phi \approx -C_n g^2\,, \tag{49}$$

where $C_1 \approx 24.5$ and $C_2 \approx 31.5$.

For large g the flux of the degree n solution tends to $-2\pi n$ for $n = 1, 2$. These are precisely the allowed values of the magnetic flux in models such as the abelian Higgs model where the flux is quantized for topological reasons. Thus, although there is no such reason for flux quantization in our model we observe an effective quantization in the strong coupling limit.

To understand the effective flux quantization and the limit of the energy as $g \to \infty$ better, we look at the dependence of the gauged baby Skyrmion's shape on the coupling g. The function a characterizing the magnetic field is plotted in Fig. 3(a) for several values of g. Note in particular that for strong coupling, a tends to a step function, taking the value 0 at the origin but -1 everywhere else. Thus at strong coupling the magnetic field is increasingly localized at the origin. This is certainly consistent with Eq. (31) in the limit of large g, although it is not obviously implied by it. Note also that $a_\infty = -1$ implies, via (44), our earlier observation that at strong coupling the magnetic flux is quantized in units of 2π.

In Fig. 3(b) we plot the profile function f for a range of couplings. We have not included more plots at weak coupling ($g < \mu$) because the profile function barely changes in this regime. Thus, as assumed in the calculation of the proton-neutron mass difference in [4], the back reaction of the electromagnetic field on the scalar field is negligible at weak coupling. At strong coupling, however, the profile function changes significantly, and the configuration becomes more localized. We conjecture that f also tends to a singular step function in the limit $g \to \infty$. To justify this conjecture, consider Eqs. (30) and (31). When a is the step function described above, these equations decouple everywhere except at the origin, and the first equation becomes the Euler-Lagrange equation derived from the functional

$$F[f] = \int dr\, r \left(\frac{1}{2} f'^2 + \mu^2 (1 - \cos f) \right). \tag{50}$$

However, this functional cannot be minimized by a non-singular function satisfying the boundary conditions (27) and (28). For suppose there were such a minimum, and call it f_1. Then define $f_\kappa(r) = f_1(r/\kappa)$, and find that $F[f_\kappa] < F[f_1]$ for $\kappa < 1$. Thus one can always lower the value of $F[f]$ by making f spikier.

To sum up, we have the following picture for the $n = 1$ and the $n = 2$ solitons in the strong coupling limit: both the magnetic field and the energy distribution become localized near the origin, tending to singular distributions as $g \to \infty$. In this limit, the total energy does not vanish because of the contributions from gradient terms in the energy density, and the magnetic flux tends to $-2\pi n$.

To end, let us look at the solutions of (30) and (31) for a particular value of g in more detail. Ideally we would like to pick a "physical" value for g which allows us to estimate the magnitude of the effects we have observed in the physically relevant (3+1)-dimensional Skyrme model. In practice it is not clear how to define "physical" here, but the following procedure should lead to a value for g which is at least in the right ball park.

The basic idea is to treat our model as if its solitons described physical baryons and its elementary quanta were physical pions: this allows us to fix the energy and length scale, and to compute a definite value for the coupling constant g from the physical value of the fine structure constant. Thus we assume, in accordance with nature, that the electromagnetic coupling constant is small.

First we fix the energy scale H by identifying the mass of a gauged baby Skyrmion with the physical nucleon mass of 940 MeV. Since for small g the mass of the gauged baby Skyrmion is virtually independent of g, we pick the value of the mass at $g = 0$; this leads to $H = 48$ MeV. To find a physical length scale λ we note that $1/\mu$ is the equivalent to the Compton wavelength of the pion in the Skyrme model. Thus we choose λ such that $1/\mu = 1.41$ MeV, i.e. $\lambda = \sqrt{0.1} \cdot 1.41\text{fm} = 0.45$ fm. To compute Planck's constant in geometric units we write

$$\hbar = 197.3 \text{ MeV fm} \approx 9.1(48\text{MeV})(0.45 \text{ fm}) \tag{51}$$

and deduce that $\hbar = 9.1$ in geometric units. Finally we use the physical value of the fine structure constant $\alpha = e^2/(4\pi\hbar) \approx 1/137$. Here e is the electron's charge which is related to the coupling constant g via $e = g\hbar$. With $\hbar = 9.1$ we conclude $g = 0.1$ in geometric units.

At this value for g the gauged baby Skyrmion is lighter than the baby Skyrmion at $g = 0$ by $\Delta E_1 = 0.12$, which is 5.9 MeV in physical units. For the solution with $n = 2$ the corresponding energy difference is $\Delta E_2 = 0.56$, which is 27 MeV in physical units. It is also interesting to look at the g-dependence of the difference between the energy of the $n = 2$ solution and twice the energy of the $n = 1$ solution, which may be interpreted as a binding energy. In the $g = 0$ case, this is about 6.6%, but it is 14% when $g = 0.1$. Thus the inclusion of the electromagnetic field leads to a more strongly bound $n = 2$ soliton.

The precise shape of the solutions for $g = 0.1$ can be seen in Fig. 4., where we plot the energy density and the magnetic field for both $n = 1$ and $n = 2$. The baby Skyrmion's energy distribution is bell-shaped and peaked at the origin, whereas the energy distribution for the soliton of degree 2 is maximal on a ring with radius $r = 1.78$.

VI Conclusions

In this paper we have studied soliton solutions of the coupled Skyrme-Maxwell system in (2+1) dimensions. The rotationally symmetric solitons we have considered necessarily carry a magnetic field but the electric field is zero. The magnetic flux can take arbitrary values, but in the strong coupling limit we observe an effective flux quantization. The soliton mass decreases when the electromagnetic coupling constant is increased and all the other parameters of the model are kept fixed. Thus a baby Skyrmion can lower its mass by interacting with the electromagnetic field.

Although the $U(1)$ gauge group is unbroken the baby Skyrmions' magnetic field is short-ranged. The reason for this is that the electromagnetic current carried by the baby Skyrmion only has a magnetic dipole component; in (2+1) dimensions static magnetic dipoles, however, have no long-range fields in Maxwellian electromagnetism. This observation has important consequences for the interaction of gauged baby Skyrmions. Since the scalar fields fall off like $\exp(-\mu r)$ and the magnetic field like $\exp(-2\mu r)$, the magnetic forces will be negligible compared to the scalar forces between two well-separated gauged baby Skyrmions. Thus, to first approximation, the forces should be the scalar dipole-dipole forces between purely scalar baby Skyrmions discussed in detail in [6].

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Figure Captions

Fig. 1: Energy "per Skyrmion" as a function of the coupling constant g for $n = 1$ and $n = 2$. The dashed line is a plot of the Bogomol'nyi bound, see (20) and (21).

Fig. 2: The magnetic flux "per Skyrmion" as a function of the coupling constant g for $n = 1$ and $n = 2$.

Fig. 3: The function a (**a**) and the profile function f (**b**) of the $n = 1$ solution for various values of the coupling constant g.

Fig. 4: The energy density (**a**) and the magnetic field (**b**) for $n = 1$ and $n = 2$ at $g = 0.1$; the function e plotted in (**a**) is the integrand of (13) divided by 4π.