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Poisson structure and symmetry in the Chern-Simons formulation of (2+1)-dimensional gravity

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Abstract

In the formulation of (2+1)-dimensional gravity as a Chern-Simons gauge theory, the phase space is the moduli space of flat Poincaré group connections. Using the combinatorial approach developed by Fock and Rosly, we give an explicit description of the phase space and its Poisson structure for the general case of a genus g oriented surface with punctures representing particles and a boundary playing the role of spatial infinity. We give a physical interpretation and explain how the degrees of freedom associated with each handle and each particle can be decoupled. The symmetry group of the theory combines an action of the mapping class group with asymptotic Poincaré transformations in a non-trivial fashion. We derive the conserved quantities associated to the latter and show that the mapping class group of the surface acts on the phase space via Poisson isomorphisms.

1 Introduction

The interest of (2+1)-dimensional gravity (two spatial dimensions, one time dimension) is that it can serve as a toy model for regular Einstein gravity in (3+1) dimensions. In the (2+1)-dimensional case, the Einstein equations of motion reduce to the requirement that the spacetime be flat outside the regions where matter is located. As a result, Einstein gravity in (2+1) dimensions is much simpler than its (3+1)-dimensional analogue. The only degrees of freedom present in (2+1)-dimensional gravity are a finite number of global degrees of freedom related to matter and the topology of the spacetime manifold. This gives rise to hope that (2+1)-dimensional gravity will allow one to study conceptual

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questions related to the quantisation of general relativity without being hindered by the technical difficulties present in the (3+1)-dimensional case.

As the features of a quantum theory are modelled after their classical counterparts and should reduce to them in the classical limit, an important prerequisite of the quantisation of (2+1) gravity is the investigation of its phase space. This includes among others the parametrisation of the phase space, the determination of its Poisson structure and the identification of symmetries and conserved quantities. Starting with the work of Regge and Nelson [1], these questions have been addressed by a number of physicists, using a variety of methods, see [2] for further background. For us, the recent work by Matschull [3] which is entirely based on the Einstein formulation of (2+1) gravity is going to be a key reference. However, despite these efforts, a complete and explicit description of the phase space and its Poisson structure for arbitrarily many (spinning) particles on a surface with handles is still missing. In view of our opening paragraph it is furthermore desirable that such a description provides a starting point for an equally explicit quantisation of the theory.

It was first noted in [4] and further discussed in [5] that the Einstein-Hilbert action in (2+1) gravity can be written as the Chern-Simons action of an appropriate gauge group. If one adopts this formulation, a large body of mathematical knowledge becomes available and can be applied to the phase space of (2+1) gravity. The phase space of a Chern-Simons theory with gauge group G is the space of flat G connections modulo gauge transformations, in the following referred to as the moduli space. It inherits a Poisson structure from the canonical Poisson structure on the space of Chern-Simons gauge fields. The moduli space of flat G connections and its Poisson structure has been investigated extensively in mathematics for the case of compact semi-simple Lie groups G. Of special interest for this article is the work of Fock and Rosly [6], in which the moduli space arises as a quotient of two finite dimensional spaces and the Poisson structure is given explicitly in terms of a classical r-matrix [6]. This description of the moduli space was developed further in [7],[8], using the language of Poisson-Lie groups. Moreover, it is the starting point for a quantisation procedure called combinatorial quantisation, carried out and described in detail in [9], [10] and [11].

The aim of this article is to apply these methods to the Chern-Simons formulation of (2+1) gravity in order to give a description of the phase space that allows for a systematic study of its physics and which is amenable to quantisation. One advantage of this approach is its generality. We are able to give an explicit parametrisation of the phase space on a spacetime manifold of topology $\mathbb{R} \times S^\infty_{g,n}$, where $S^\infty_{g,n}$ is a genus g oriented surface with punctures representing n massive particles (possibly with spin) and a connected boundary corresponding to spatial infinity. In this description, the mathematical concepts and parameters introduced in [6] have a natural physical interpretation. We thus obtain a mathematically rigorous framework in which the physics questions can be addressed.

Our article is structured as follows. Sect. 2 contains an introduction to the Chern-Simons formulation of (2+1) gravity and its phase space as the moduli space of flat connections. We summarise briefly some mathematical results about the moduli space, in particular the work of Fock and Rosly [6] and Schomerus and Alekseev [11] underlying our description of the phase space.

In Sect. 3 we apply these results to the phase space of (2+1) gravity on an open surface $S^{\infty}_{0,1}$ with a single massive particle. We extend the description of the moduli space given by Fock and Rosly to incorporate a boundary representing spatial infinity. This allows us to give an explicit parametrisation of the phase space and its Poisson structure as well as a physical interpretation. Relating the Poisson structure of the one particle phase space to the symplectic structure on the dual of the universal cover of the (2+1)-dimensional Poincaré group, we show that it generalises the Poisson structure derived by Matschull and Welling [12] in the metric formulation.

Sect. 4 extends these results to the general case of a genus g oriented surface with n punctures and a boundary representing spatial infinity. After a discussion of the boundary condition and asymptotically nontrivial gauge transformations, we describe the phase space by means of a graph and derive its Poisson structure. The results are given a physical interpretation and related to the theory of Poisson-Lie groups. Applying the work of Alekseev and Malkin [7],[8], we introduce a set of normal coordinates that decouple the contributions of different handles and particles.

In Sect. 5, we study the symmetries of the phase space. Following the work of Giulini [13], we identify the symmetry group of (2+1) gravity in the Chern-Simons formulation for a spacetime of topology $\mathbb{R} \times S^{\infty}_{g,n}$. We determine the quantities that generate asymptotic symmetries via the Poisson bracket and interpret them in physical terms. The action of large diffeomorphisms on the spatial surface $S^{\infty}_{g,n}$ is investigated, and we prove that the generators of the (full) mapping class group act on the phase space as Poisson isomorphisms.

Our final section contains comments on the relationship between our results and other approaches to (2+1)-dimensional gravity, our conclusions and an outlook. Facts and definitions about the universal cover of the (2+1)-dimensional Poincaré group as a Poisson-Lie group are summarised in the appendix.

2 The Chern-Simons formulation of (2+1)-dimensional gravity and its phase space as the moduli space of flat connections

2.1 Conventions

In the following we restrict attention to (2+1)-dimensional gravity with vanishing cosmological constant in its formulation as a Chern-Simons theory on a spacetime M of topology $M \approx \mathbb{R} \times S^{\infty}_{g,n}$, where $S^{\infty}_{g,n} = S^{\infty}_{g} - \{z_1, \ldots, z_n\}$ is a disc with n punctures and g handles, i.e. an oriented surface of genus g with n punctures and a connected boundary. The punctures at $z_1, \ldots, z_n$ represent massive particles, the boundary corresponds to spatial infinity.

Throughout the paper we use units in which the speed of light is 1. Exploiting the fact that in (2+1) gravity Newton's constant has dimensions of inverse mass, we measure masses in units of $(8\pi G)^{-1}$. Indices are raised and lowered with the (2+1)-dimensional

Minkowski metric η_{ab} =diag$(1,-1,-1)$, and Einstein summation convention is employed unless stated otherwise. For the epsilon tensor we choose the convention $\epsilon_{012}=1$.

$L_3^\uparrow$ and $P_3^\uparrow = L_3^\uparrow \ltimes \mathbb{R}^3$ denote respectively the (2+1)-dimensional proper ortochronous Lorentz and Poincaré group and $\tilde{L}_3^\uparrow$, $\tilde{P}_3^\uparrow = \tilde{L}_3^\uparrow \ltimes \mathbb{R}^3$ their universal covers. The Lie algebra of the group $\tilde{P}_3^\uparrow$ is Lie $\tilde{P}_3^\uparrow = iso(2,1)$ with generators P_a, J_a, $a=0,1,2$, and the commutator

$$[P_a, P_b] = 0, \quad [J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, P_b] = \epsilon_{abc} P^c. \tag{2.1}$$

The generators J_a, $a=0,1,2$, span the Lie algebra $so(2,1)$ of $\tilde{L}_3^\uparrow$. If we write the elements of $\tilde{P}_3^\uparrow$ as

$$(u, \boldsymbol{a}) \in \tilde{P}_3^\uparrow \qquad \text{with} \qquad u \in \tilde{L}_3^\uparrow,\ \boldsymbol{a} \in \mathbb{R}^3,$$

the group multiplication in $\tilde{P}_3^\uparrow$ is given by

$$(u_1, \boldsymbol{a}_1) \cdot (u_2, \boldsymbol{a}_2) = (u_1 \cdot u_2, \boldsymbol{a}_1 + \mathrm{Ad}(u_1)\boldsymbol{a}_2) \tag{2.2}$$

with $\mathrm{Ad}(u)$ denoting the $L_3^\uparrow$ element associated to $u \in \tilde{L}_3^\uparrow$.

In the following we adopt the conventions of [14] for parametrising the (2+1)-dimensional Lorentz group and its covers, in particular the parametrisation via the exponential map

$$\exp : so(2,1) \to \tilde{L}_3^\uparrow. \tag{2.3}$$

Note that this map is neither into nor onto, but that we can nevertheless write any element $u \in \tilde{L}_3^\uparrow$ in the form

$$u = \exp(-2\pi n J_0) \exp(-p^a J_a) \qquad \text{with} \qquad n \in \mathbb{Z}. \tag{2.4}$$

Combining the parameters p^a into a three-vector $\boldsymbol{p} = (p^0, p^1, p^2)$, we can characterise elliptic elements of $\tilde{L}_3^\uparrow$ by the condition $\boldsymbol{p}^2 = p_a p^a \in (0, (2\pi)^2)$, parabolic elements by $\boldsymbol{p}^2 = 0$ and hyperbolic elements by $\boldsymbol{p}^2 < 0$.

This allows us to express the adjoint $\mathrm{Ad}(u)$ of an element $u \in \tilde{L}_3^\uparrow$ as

$$\begin{aligned}\mathrm{Ad}(u)_{ab} &= \delta_{ab} + (\delta_{ab} - p_a p_b)\left(\sum_{k=1}^{\infty} \frac{(-1)^k (\boldsymbol{p}^2)^{k-1}}{(2k)!}\right) + \epsilon_{abc} p^c \left(\sum_{k=0}^{\infty} \frac{(-1)^k (\boldsymbol{p}^2)^k}{(2k+1)!}\right) \\ &= \begin{cases} \hat{p}_a \hat{p}_b + \cos(\sqrt{\boldsymbol{p}^2})\,(\delta_{ab} - \hat{p}_a \hat{p}_b) + \sin(\sqrt{\boldsymbol{p}^2})\epsilon_{abc}\hat{p}^c & \text{for } u \text{ elliptic} \\ \delta_{ab} + \epsilon_{abc} p^c + \frac{1}{2} p_a p_b & \text{for } u \text{ parabolic} \\ \hat{p}_a \hat{p}_b + \cosh(\sqrt{|\boldsymbol{p}^2|})\,(\delta_{ab} - \hat{p}_a \hat{p}_b) + \sinh(\sqrt{|\boldsymbol{p}^2|})\epsilon_{abc}\hat{p}^c & \text{for } u \text{ hyperbolic,} \end{cases} .\end{aligned} \tag{2.5}$$

where $\hat{\boldsymbol{p}} = \boldsymbol{p}/\sqrt{|\boldsymbol{p}^2|}$. Elliptic conjugacy classes in the group $\tilde{P}_3^\uparrow$ are characterised by two restrictions on the parameter three-vectors $\boldsymbol{p}$, $\boldsymbol{a}$

$$\boldsymbol{p}^2 = \mu^2 \qquad \boldsymbol{p}\cdot\boldsymbol{j} = \mu s \qquad \text{where} \qquad \boldsymbol{j} := -\mathrm{Ad}(u^{-1})\boldsymbol{a} \tag{2.6}$$

with parameters $\mu \in (0, 2\pi)$, $s \in \mathbb{R}$.

2.2 (2+1)-dimensional gravity as a Chern-Simons gauge theory

In Einstein's original formulation of general relativity, the dynamical variable is a metric g on M. For the Chern-Simons formulation it is essential to adopt Cartan's point of view, where the theory is formulated in terms of the (non-degenerate) dreibein of one-forms e_a, $a = 0, 1, 2$, and the spin connection one-forms ω_a, $a = 0, 1, 2$. The dreibein is related to the metric via

$$\eta^{ab} e_a \otimes e_b = g, \tag{2.7}$$

and the one-forms ω_a should be thought of as components of the $\tilde{L}_3^{\uparrow}$ connection

$$\omega = \omega_a J^a. \tag{2.8}$$

The Cartan formulation is equivalent to Einstein's metric formulation of (2+1) gravity provided that the dreibein is invertible.

The vacuum Einstein-Hilbert action in (2+1) dimensions can be written in terms of dreibein and spin connection as

$$S_{EH}[\omega, e] = \int_M e_a \wedge F_\omega^a, \tag{2.9}$$

where F_ω^a denotes the components of the curvature two-form:

$$F_\omega = d\omega + \frac{1}{2}[\omega, \omega] = F_\omega^a J_a, \qquad\qquad F_\omega^a = d\omega^a + \frac{1}{2}\epsilon^a_{\ bc}\omega^b \wedge \omega^c. \tag{2.10}$$

Both the connection ω_a and the dreibein e_a are dynamical variables and varied independently. Variation with respect to the spin connection yields the requirement that torsion vanishes:

$$D_\omega e_a = de_a + \epsilon_{abc}\omega^b e^c = 0. \tag{2.11}$$

Variation with respect to e_a yields the vanishing of the curvature tensor:

$$F_\omega = 0. \tag{2.12}$$

In (2+1) dimensions, this is equivalent to the Einstein equations in the absence of matter.

For the Chern-Simons formulation of gravity, dreibein and the spin connection are combined into a Cartan connection [15]. This is a one-form with values in the Lie algebra $iso(2, 1)$

$$A = \omega_a J^a + e_a P^a, \tag{2.13}$$

whose curvature

$$F = (D_\omega e^a)P_a + (F_\omega^a)J_a \tag{2.14}$$

combines the curvature and the torsion of the spin connection.

The final ingredient needed to establish the Chern-Simons formulation is a non-degenerate, invariant bilinear form on the Lie algebra $iso(2,1)$

$$\langle J_a, P_b\rangle = \eta_{ab}, \quad \langle J_a, J_b\rangle = \langle P_a, P_b\rangle = 0. \tag{2.15}$$

Then the Chern-Simons action for the connection A on M is

$$S_{CS}[A] = \frac{1}{2}\int_M \langle A\wedge dA\rangle + \frac{2}{3}\langle A\wedge A\wedge A\rangle. \tag{2.16}$$

A short calculation shows that this is equal to the Einstein-Hilbert action (2.9). Moreover, the equation of motion found by varying the action with respect to A is

$$F = 0. \tag{2.17}$$

Using the decomposition (2.14) we thus reproduce the condition of vanishing torsion and the three-dimensional Einstein equations, as required. This shows that there is a one-to-one correspondence of solutions of Einstein's equations and flat Chern-Simons gauge fields with non-degenerate dreibein.

For a spacetime of the form $S_g^\infty - \{z_1,\ldots,z_n\}$ with punctures corresponding to massive particles, an additional condition has to be given regarding the behaviour of the curvature tensor at the punctures. The inclusion of massive particles with spin into (2+1)-dimensional gravity in its Chern-Simons formulation has been investigated by several authors. As explained in [16], the curvature tensor develops δ-function singularities at the positions of the particles. For a single particle of mass μ and spin s at rest at z_1 the curvature tensor is given by

$$F(z) + (\mu J_0 + sP_0)\,\delta(z - z_1) = 0 \tag{2.18}$$

As a consequence, the holonomy for an infinitesimal circle surrounding the particle is

$$h_0 = (e^{-\mu J_0}, -(s,0,0)^t). \tag{2.19}$$

As we will show in Sect. 3, the holonomy for a general loop around the particle is related to h_0 via conjugation with a $\tilde{P}_3^\uparrow$-element. It is an element of a fixed elliptic $\tilde{P}_3^\uparrow$-conjugacy class parametrised by the particle's mass μ and spin s as in (2.6).

2.3 The phase space of Chern-Simons theory

In a Chern-Simons theory with gauge group G on manifold $M = \mathbb{R}\times S$, where S is a two-dimensional surface with connected boundary, two gauge connections A and A' describe the same physical state if they are related by a gauge transformation

$$A' = \gamma A\gamma^{-1} + \gamma d\gamma^{-1} \tag{2.20}$$

where γ is a G-valued function on the surface S. It must satisfy an appropriate fall-off condition compatible with the conditions imposed on the gauge connections at the boundary. The phase space of a Chern-Simons theory with gauge group G on a manifold M

is the space of all physically distinct solutions of the equations of motion (2.17) subject to conditions of the form (2.18) at the punctures. It is the moduli space of flat G connections on S modulo gauge transformations (2.20). It inherits a Poisson structure from the canonical symplectic structure on the space of gauge connections [17]. The properties of the moduli space have been investigated extensively in mathematics for the case of compact gauge groups G. In particular, it has been shown that it is finite dimensional. A review of the mathematical results and further references are given in the book [18].

The Fock-Rosly description of the moduli space

Fock and Rosly gave a description of the moduli space of flat G connections on a closed, oriented surface S with punctures by means of a graph embedded into the surface [6], see[19] for a pedagogical account. The underlying idea is similar to lattice gauge theory, but - due to the absence of local degrees of freedom - the physical content of the theory is captured entirely by a sufficiently refined[3] graph without the need to take a continuum limit. According to Fock and Rosly, the moduli space and its Poisson structure on an oriented, surface with punctures can be uniquely characterised by a *ciliated fat graph* Γ. This is a set of N_Γ vertices and I_Γ oriented edges connecting the vertices together with a linear ordering of the incident edges at each vertex. If such a graph is embedded into the surface S, the orientation induces a cyclic ordering of the incident edges at each vertex and makes the graph a *fat graph*. The surface can be completely reconstructed from a sufficiently refined fat graph. The reconstruction of the surface and the Poisson structure on the moduli space requires a *ciliated fat graph*. It can be obtained from a fat graph embedded into the surface by adding a cilium at each vertex in order to separate the incident edges of minimum and maximum order. Given a smooth G connection on the surface S and a ciliated fat graph Γ embedded into it, parallel transport along the edges of the graph assigns an element A_i of the gauge group to each oriented edge i and thus induces a *graph connection*: a map from the set of oriented edges into the direct product of I_Γ copies of the gauge group G. Flat connections on the surface induce *flat graph connections*, for which the ordered product of the group elements assigned to edges around a face of the graph is trivial if the face does not contain any punctures. Similarly, gauge transformations (2.20) induce transformations of the group elements associated to each edge: The group element A_i assigned to the oriented edge i by parallel transport transforms according to

$$A_i \to \gamma(z([i^\vee]))A_i\gamma^{-1}(z([i])), \tag{2.21}$$

where $z([i])$ denotes the position of the vertex edge i points to and $z([i^\vee])$ the position of the vertex at which it starts. This defines a *graph gauge transformation*, a map from the set of vertices into the direct product of N_Γ copies of the gauge group. As explained in [6], for any sufficiently refined graph Γ the moduli space on S is isomorphic the quotient of the space $\mathcal{A}_\Gamma$ of flat graph connections modulo graph gauge transformations $\mathcal{G}_\Gamma$.

[3] As we use only sufficiently refined graphs in this article, we will not explain this concept further.

Theorem 2.1 *The moduli space $\mathcal{M}$ on a closed, oriented surface S with punctures is isomorphic to the quotient of the space of graph connections modulo graph gauge transformations for any sufficiently refined fat graph Γ*

$$\mathcal{M} \cong \mathcal{A}_\Gamma / \mathcal{G}_\Gamma.$$

This characterises the moduli space in terms of two *finite* dimensional spaces. The essential advantage of the approach by Fock and Rosly is that it allows one to express the Poisson structure on the moduli space by a Poisson structure defined on the space $\mathcal{A}_\Gamma$ of graph connections rather than the Poisson structure on the (infinite dimensional) space of gauge connections. The construction of this Poisson structure involves the assignment of a classical r-matrix for the Lie algebra $\mathfrak{g}$ to each vertex of the graph. This is an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ which satisfies the classical Yang-Baxter equation (A.4) and whose symmetric part is equal to the tensor representing a non-degenerate invariant bilinear form on $\mathfrak{g}$; details are given in the appendix.

Theorem 2.2 *(Fock, Rosly)*

1. *Let Γ be a sufficiently refined ciliated fat graph. Assign a r-matrix to each vertex ν, and let $r^{\alpha\beta}(\nu)$ be its components with respect to a basis $\{X_\alpha\}$, $\alpha = 1, ..., dim\,\mathfrak{g}$ of $\mathfrak{g}$. Then the following bivector defines a Poisson structure on the space of graph connections $\mathcal{A}_\Gamma$*

$$\begin{aligned} B_{FR} \;\; = & \sum_{vertices\,\nu=1}^{N_\Gamma} \left(\frac{1}{2} \sum_{i\in\nu} r^{\alpha\beta}(\nu) R^i_\alpha \wedge R^i_\beta + \frac{1}{2} \sum_{i^\vee\in\nu} r^{\alpha\beta}(\nu) L^i_\alpha \wedge L^i_\beta \right. \\ + & \sum_{i,j\in\nu, i<j} r^{\alpha\beta}(\nu) R^i_\alpha \wedge R^j_\beta + \sum_{i,j^\vee\in\nu, i<j^\vee} r^{\alpha\beta}(\nu) R^i_\alpha \wedge L^j_\beta \\ + & \sum_{i^\vee,j\in\nu, i^\vee<j} r^{\alpha\beta}(\nu) L^i_\alpha \wedge R^j_\beta + \sum_{i^\vee,j^\vee\in\nu, i^\vee<j^\vee} r^{\alpha\beta}(\nu) L^i_\alpha \wedge L^j_\beta \\ + & \left. \sum_{i,i^\vee\in\nu} r^{\alpha\beta}(\nu) R^i_\alpha \wedge L^i_\beta \right), \end{aligned} \tag{2.22}$$

   where $i \in \nu$ denotes an edge pointing towards vertex ν, $i^\vee \in \nu$ an edge starting at vertex ν. R^i_α and L^i_α, respectively, are the left and right invariant vector fields associated to edge i with respect to the basis X_α of $\mathfrak{g}$ and $>$, $<$ refer to the ordering of the incident edges at each vertex. The convention $i < i^\vee$ is chosen for each edge starting and ending at the same vertex.

2. *Let $\mathcal{G}_\Gamma \cong G^{N_\Gamma}$ be equipped with the N_Γ-fold direct product of the Poisson structure on the group G that is defined by means of a r-matrix as described in the appendix. The elements of the group $\mathcal{G}_\Gamma$ of graph gauge transformations act as Poisson maps $\mathcal{G}_\Gamma \times \mathcal{A}_\Gamma \to \mathcal{A}_\Gamma$ with respect to the direct product Poisson structure on $\mathcal{G}_\Gamma \times \mathcal{A}_\Gamma$ and the Poisson structure on $\mathcal{A}_\Gamma$ defined by (2.22).*

3. *As graph gauge transformations are Poisson maps, the Poisson structure defined by* (2.22) *induces a Poisson structure on the quotient* $\mathcal{M}$. *It is independent of the graph and isomorphic to the Poisson structure induced by the canonical symplectic structure on the space of gauge connections.*

The Poisson bivector (2.22) can be decomposed into a part tangential to the gauge orbits and a part transversal to them:

$$B_{FR} = \Big(\sum_{\text{vertices } \nu=1}^{N_\Gamma} r^{\alpha\beta}_{(a)}(\nu) \big(\sum_{i\in\nu} R^i_\alpha + \sum_{i^\vee\in\nu} L^i_\alpha \big) \otimes \big(\sum_{j\in\nu} R^j_\beta + \sum_{j^\vee\in\nu} L^j_\beta \big) \Big) \tag{2.23}$$
$$+ \Big(\sum_{\text{vertices } \nu=1}^{N_\Gamma} t^{\alpha\beta} \sum_{i,j\in\nu, i<j} R^i_\alpha \wedge R^j_\beta + \sum_{i,j^\vee\in\nu, i<j^\vee} R^i_\alpha \wedge L^j_\beta + \sum_{i^\vee,j\in\nu, i^\vee<j} L^i_\alpha \wedge R^j_\beta$$
$$+ \sum_{i^\vee,j^\vee\in\nu, i^\vee<j^\vee} L^i_\alpha \wedge L^j_\beta + \sum_{i,i^\vee\in\nu} R^i_\alpha \wedge L^i_\beta \Big) .$$

The first line in formula (2.23) depends only on the antisymmetric parts

$$r^{\alpha\beta}_{(a)}(\nu) = \tfrac{1}{2}(r^{\alpha\beta}(\nu) - r^{\beta\alpha}(\nu)) \tag{2.24}$$

of the r-matrices assigned to each vertex and is tangential to the gauge orbits. The second part of the bivector and with it the Poisson structure on the moduli space depends only on the symmetric part common to all r-matrices, the components

$$t^{\alpha\beta} = \tfrac{1}{2}(r^{\alpha\beta}(\nu) + r^{\beta\alpha}(\nu)) \tag{2.25}$$

of the matrix representing the bilinear form on the Lie algebra $\mathfrak{g}$. In particular, this implies that the Poisson structure on the moduli space is not affected by the choice of the r-matrix at each vertex.

The description using the fundamental group

Alekseev, Grosse and Schomerus specialised this description of the moduli space to the simplest graph that can be used to characterise a closed surface with punctures: a choice of generators of its fundamental group [9], [10] [11]. For a closed surface S of genus $g \geq 0$ with $n \geq 0$ punctures, the fundamental group $\pi_1(z_0, S)$ with respect to a basepoint z_0 is generated by $2g+n$ curves starting and ending at z_0, two curves a_j, b_j, $j = 1, \ldots, g$, around each handle and a loop m_i, $i = 1, \ldots, n$, around each puncture (see Fig. 1).

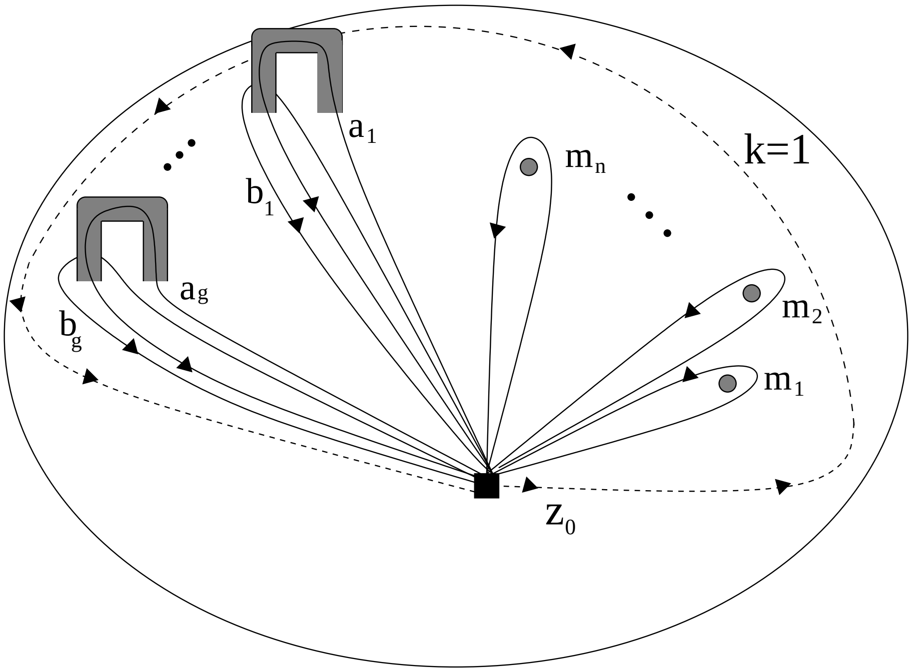

Fig. 1

The generators of the fundamental group on a genus g surface with n punctures

Its generators obey a single relation

$$k = [b_g, a_g^{-1}] \cdot \ldots \cdot [b_1, a_1^{-1}] \cdot m_n \cdot \ldots \cdot m_1 = 1, \tag{2.26}$$

where $[\,,\,]$ is the group commutator $[x, y] = xyx^{-1}y^{-1}$. Via the holonomy, a graph connection assigns an element of the gauge group G to each of the generators. Due to equation (2.18), the holonomies M_i of the loops m_i associated to the punctures must be elements of a fixed conjugacy classes $\mathcal{C}_i$, whereas there is no such constraint for the group elements A_j, B_j associated to the curves around the handles. Taking this into account, the space of graph connections is given as

$$\begin{aligned} \mathcal{A}_{\pi_1} &= \{\phi \in \mathrm{Hom}(\pi_1(z_\infty, S_{g,n}), G) \mid M_i = \phi(m_i) \in \mathcal{C}_i, i = 1, \ldots, n\} \qquad (2.27) \\ &\cong \{(M_1, \ldots, B_g) \in G^{2g+n} \mid [B_g, A_g^{-1}] \cdot \ldots \cdot [B_1, A_1^{-1}] \cdot M_n \cdot \ldots \cdot M_1 = 1, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad M_i \in \mathcal{C}_i, i = 1, \ldots, n\}. \end{aligned}$$

As the graph defined by the fundamental group has only one vertex, the group of graph gauge transformations is the gauge group $\mathcal{G}_{\pi_1} = G$, acting on the holonomies by global conjugation, and the moduli space is the quotient

$$\mathcal{M} = \mathcal{A}_{\pi_1}/G. \tag{2.28}$$

The Poisson bivector (2.22) defines a Poisson structure on the space A_{π_1} of graph connections that induces a Poisson structure equivalent to the canonical Poisson structure on the moduli space.

2.4 The phase space of (2+1) gravity as the moduli space of flat $\tilde{P}_3^{\uparrow}$ connections

The material of the previous two subsections allows us to formulate in precise, technical terms the questions we have to address when applying the Fock-Rosly description of the moduli space to the Chern-Simons formulation of (2+1)-dimensional gravity.

The first ingredient required to implement the Fock-Rosly description is a classical r-matrix for the Lie algebra $iso(2,1)$ of the Poincaré group. As we shall see in the next section, the required r-matrix corresponds to the one given in [20] for the Euclidean case and can easily be adapted to the Lorentzian setting.

Second, we have to choose the gauge group G. The Chern-Simons action only depend on the Lie algebra $iso(2,1)$, and *a priori* we are free to consider the Poincaré group or one of its covers as the gauge group. In this paper we take the gauge group G to be the universal cover $\tilde{P}_3^{\uparrow}$, as this leads to various technical simplifications.

Since essentially all mathematical work on the moduli space of flat connections has been carried out in the context of compact gauge groups one might worry about complications caused by working with the non-compact group $\tilde{P}_3^{\uparrow}$. However, the difficulties associated to non-compact gauge groups arise when taking quotients by an action of the gauge group. It turns out that in our application to (2+1) gravity, such quotients never arise. This is due to the fact that we work with a surface with boundary and thus closely related to the next challenge, namely the incorporation of a surface with boundary in the Fock-Rosly formalism. The details of how this is done are the subject of the next section. However, one consequence is that we do not need to take the quotient (2.28).

The final issue to be investigated is that of gauge invariance and symmetry. The Chern-Simons formulation of (2+1)-dimensional gravity is invariant under Chern-Simons gauge transformations as well as under diffeomorphisms. However, large diffeomorphisms (not connected to the identity) and asymptotically non-trivial gauge transformations do not relate physically equivalent configurations. They combine in a rather subtle way to form the symmetry group of the theory.

3 The phase space for a single massive particle

3.1 The open spacetime containing a single particle

We begin our investigation of the phase space of (2+1)-dimensional gravity with the case of an open universe of topology $\mathbb{R} \times S^{\infty}_{0,1}$, containing a single massive particle. In the metric formalism, a spacetime with a particle of mass μ and spin s that is at rest at the origin, is described by the conical metric [21]

$$ds^2 = (dt + s\, d\varphi)^2 - \frac{1}{(1-\frac{\mu}{2\pi})^2} dr^2 - r^2 d\varphi^2 \quad \text{with } r \in (0,\infty),\ t \in \mathbb{R},\ \varphi \in [0, 2\pi), \quad (3.1)$$

where the mass of the particle is restricted to the interval $\mu \in (0, 2\pi)$. Deficit angle $\Delta\varphi$ and time shift Δt of the cone are given by

$$\Delta\varphi = \mu \qquad \text{and} \qquad \Delta t = 2\pi s. \tag{3.2}$$

A dreibein and spin connection leading to this metric via (2.7) are

$$\begin{aligned} e^0 &= dt + s\,d\varphi & \omega^0 &= \tfrac{\mu}{2\pi} d\varphi \\ e^1 &= \frac{1}{1-\frac{\mu}{2\pi}} \cos\varphi\, dr - r \sin\varphi\, d\varphi & \omega^1 &= 0 \\ e^2 &= \frac{1}{1-\frac{\mu}{2\pi}} \sin\varphi\, dr + r \cos\varphi\, d\varphi & \omega^2 &= 0. \end{aligned} \tag{3.3}$$

With the conventions given in Sect. 2, they can be combined into a Chern-Simons gauge field

$$A_\infty = \omega^a J_a + e^a P_a. \tag{3.4}$$

As the spatial surface $S^\infty_{0,1}$ has a boundary representing spatial infinity, we must impose an appropriate boundary condition on the gauge field and restrict to gauge transformations which are compatible with this condition. We derive the boundary condition in the Chern-Simons formulation from a corresponding boundary condition in the metric formalism. In a reference frame where the massive particle is at rest at the origin, the particle's centre of mass frame) the boundary condition in the metric formalism is the (trivial) condition that the metric be conical of the form (3.1). The gauge transformations compatible with this condition must reduce to spatial rotations and translations in the time direction outside an open region containing the particle.

In the Chern-Simons formulation of (2+1)-dimensional gravity, the metric (3.1) corresponds to a gauge field (3.4) with e, ω given by (3.3). The admissible gauge transformations are asymptotically constant Chern-Simons gauge transformations. Spatial rotations are implemented by Chern-Simons gauge transformations (2.20) of A_∞, where γ is constant and takes the value

$$g = (\exp(-\theta J_0), 0)$$

with θ constant outside an open region containing the particle. For time translations we take

$$g = (1, (q_0, 0, 0)^t)$$

with q_0 constant outside an open region containing the particle. These asymptotically nontrivial transformations differ in their physical interpretation from regular gauge transformations which vanish outside a region containing the particle. They are not related to gauge degrees of freedom but physically meaningful transformations acting on the phase space. We return to this question in the next section when we consider a universe containing an arbitrary number of particles.

Note, however, that in the Chern-Simons formulation it is not necessary to limit oneself to the particle's centre of mass frame. In order to study the dynamics of a single particle we should admit general Poincaré transformations with respect to the centre of mass frame. Then the admissible transformations are Chern-Simons transformations of the form (2.20) with $d\gamma = 0$ outside an open region containing the particle. The corresponding boundary condition on the gauge field is the requirement that the gauge field be obtained from A_∞ (3.4) by an asymptotically constant gauge transformation $g = (\Lambda, \boldsymbol{q})$. i.e. that it be of the form $\mathrm{Ad}(g)A_\infty$.

3.2 Phase space and Poisson structure

The simplest graph describing the open spacetime containing a single consists of a single vertex z_∞ at the boundary and a loop around the particle. The loop can be built up from an edge connecting vertex and particle and an (infinitesimal) circle around the particle as pictured in Fig. 2.

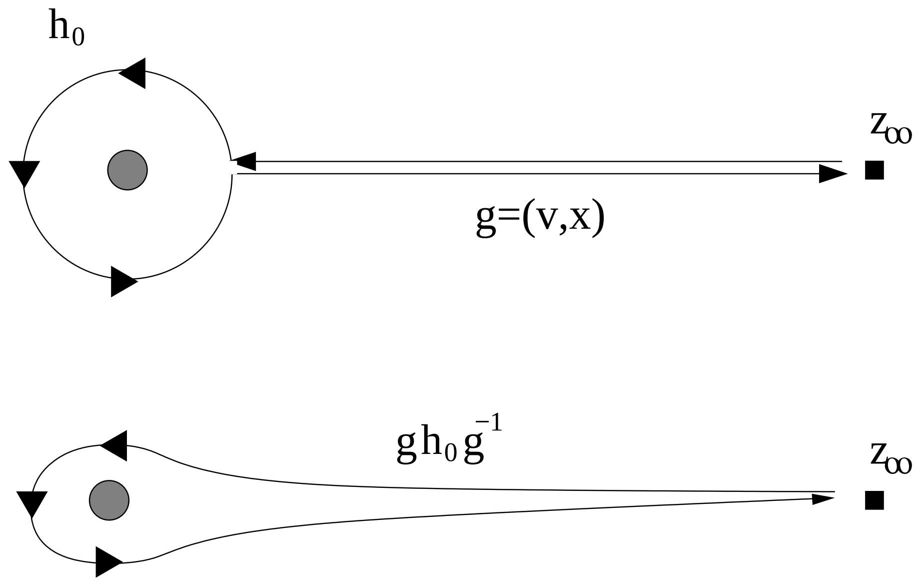

Fig. 2

The holonomy of a loop around a puncture

Using expression (3.3), we calculate that the holonomy around the infinitesimal circle is the element h_0 given by (2.19). If we write $g = (v, \boldsymbol{x})$ for the group element obtained by parallel transport along the edge connecting vertex and particle, the holonomy h of the loop starting and ending at z_∞ is

$$h = gh_0g^{-1}. \tag{3.5}$$

Defining

$$p^aJ_a = \mu\mathrm{Ad}(v)J_0 \tag{3.6}$$

and

$$u = v\exp(-\mu J_0)v^{-1} = \exp(-p^a J_a) \tag{3.7}$$

we can also write the holonomy as

$$h = (u, \boldsymbol{a}) = (u, -\mathrm{Ad}(u)\boldsymbol{j}), \tag{3.8}$$

where

$$\boldsymbol{j} = (1 - \mathrm{Ad}(u^{-1}))\boldsymbol{x} + s\hat{\boldsymbol{p}}. \tag{3.9}$$

The fact that the holonomy is an element of a fixed $\tilde{P}_3^\uparrow$-conjugacy class determined by mass μ and spin s of the particles results in constraints on the parameters p_a and j_a

$$\boldsymbol{p}^2 = p_a p^a = \mu^2 \qquad \boldsymbol{p}\cdot\boldsymbol{j} = p_a j^a = \mu s. \tag{3.10}$$

Graph gauge transformations arise from asymptotically nontrivial gauge transformations (2.20) and act on the graph by conjugating the holonomy. The moduli space on the surface would be obtained by dividing this transformations out. However, as asymptotically nontrivial transformations are physically meaningful, we do not want to do this. The extended phase space is then simply the group $\tilde{P}_3^\uparrow$ parametrised by the parameter three-vectors $\boldsymbol{p}$ and $\boldsymbol{j}$. The physical phase space is obtained from it by imposing the constraints (3.10), which selects the $\tilde{P}_3^\uparrow$-conjugacy class given by (3.5).

The Poisson structure on the phase space is defined by the Fock-Rosly bivector (2.22). For the graph introduced above it is given by

$$B_{FR} = \tfrac{1}{2} r^{\alpha\beta} R_\alpha \wedge R_\beta + \tfrac{1}{2} r^{\alpha\beta} L_\alpha \wedge L_\beta + r^{\alpha\beta} R_\alpha \wedge L_\beta. \tag{3.11}$$

As we do not divide by graph gauge transformations, the choice of the r-matrix is no longer arbitrary and different r-matrices lead to different Poisson structures on the phase space. The correct choice of the r-matrix has to be determined from physical considerations: the components of the vectors $\boldsymbol{p}$, $\boldsymbol{j}$ should have the Poisson brackets expected for momentum and angular momentum three-vector of a free relativistic particle and act as infinitesimal generators of asymptotic Poincaré transformations. As it turns out, this requires the r-matrix

$$r = P_a \otimes J^a. \tag{3.12}$$

The right and left invariant vector fields on $\tilde{P}_3^\uparrow$ are given by

$$\begin{aligned}
P^R_a f(h) &:= \frac{d}{dt}|_{t=0} f(h e^{tP_a}) = -\frac{\partial f}{\partial j^a}(h) \\
P^L_a f(h) &:= \frac{d}{dt}|_{t=0} f(e^{-tP_a} h) = \mathrm{Ad}(u)_{ab} \frac{\partial f}{\partial j_b}(h) \\
J^R_a f(h) &:= \frac{d}{dt}|_{t=0} f(h e^{tJ_a}) = \Big(\frac{1}{\mathrm{Ad}(u) - 1}\Big)_{ab} \epsilon^b{}_{cd} p^c \frac{\partial f}{\partial p^d}(h) - \epsilon_{abc} j^b \frac{\partial f}{\partial j_c}(h) \\
J^L_a f(h) &:= \frac{d}{dt}|_{t=0} f(e^{-tJ_a} h) = \Big(\frac{\mathrm{Ad}(u)}{1 - \mathrm{Ad}(u)}\Big)_{ab} \epsilon^b{}_{cd} p^c \frac{\partial f}{\partial p^d}(h),
\end{aligned} \tag{3.13}$$

where all functions are evaluated at h parametrised in terms of $\boldsymbol{p}$ and $\boldsymbol{j}$ as in (3.8)[4]. The expression $\big(\mathrm{Ad}(u)/(1-\mathrm{Ad}(u))\big)_{ab}\epsilon^b{}_{cd}p^c$ in (3.13) is defined properly by

$$\Big(\frac{\mathrm{Ad}(u)}{1-\mathrm{Ad}(u)}\Big)_{ab}\epsilon^b{}_{cd}p^c = \Pi + \frac{\frac{\mu}{2}}{\sin(\frac{\mu}{2})}\exp(-\frac{1}{2}p^a J_a)(1-\Pi), \tag{3.14}$$

where Π denotes the projector onto the direction of the momentum $\Pi_a{}^b = \hat{p}_a\hat{p}^b$. Note that there is still a coordinate singularity in the limit $\mu \to 2\pi$, where the coordinates (3.6) no longer provide a good parametrisation of the holonomy h.

Inserting the vector fields (3.13) and the r-matrix into equations (3.11), we obtain the Poisson structure in terms of the parameters p_a and j_a, $a = 0, 1, 2$,

$$\{j_a, j_b\} = -\epsilon_{abc}j^c \qquad \{j_a, p_b\} = -\epsilon_{abc}p^c \qquad \{p_a, p_b\} = 0. \tag{3.15}$$

These equations together with the constraints (3.10) suggest an interpretation of the parameters p_a as the components of the particle's three-momentum, p^0 as its energy and the spatial components p^1, p^2 as its momentum. Similarly, the vector $\boldsymbol{j}$, in the following referred to as angular momentum three-vector, contains the particle's angular momentum j^0 and the spatial components j_1, j_2 associated to the Lorentz boosts. It can be seen immediately that the constraints (3.10) are Casimir functions of this Poisson structure. Their values parametrise its symplectic leaves, the conjugacy classes in $\tilde{P}_3^\uparrow$.

3.3 Physical Interpretation

The components of the graph used to construct the Poisson structure can be given a physical interpretation as follows. The vertex z_∞ represents an observer. The $\tilde{P}_3^\uparrow$-element $g = (v, \boldsymbol{x})$ assigned to the edge connecting it with the particle gives the Poincaré transformation relating the observer's reference frame to centre of mass frame of the particle. The vector $\boldsymbol{x}$ determines the translation in time and the position, v a spatial rotation and/or Lorentz boosts.

The non-standard relation (3.9) between the angular momentum three-vector $\boldsymbol{j}$, the momentum $\boldsymbol{p}$ and the position three-vector $\boldsymbol{x}$ was first discussed in detail for the case of spin-less particles in [12] , where it was derived entirely in terms of the metric formulation of (2+1) gravity. In that paper the authors also pointed out that it leads to non-standard commutation relations of the position coordinates. In order to understand this non-commutativity from our point of view, recall that the usual (non-gravitational) formula for the angular momentum three-vector of a free relativistic particle is obtained using the co-adjoint action (instead of the conjugation action above)

$$p^a J_a + k^a P_a = g(\mu J_0 + sP_0)g^{-1}, \tag{3.16}$$

leading to the familiar formula for the angular momentum three-vector $\boldsymbol{k}$

$$\boldsymbol{k} = \boldsymbol{x} \wedge \boldsymbol{p} + s\hat{\boldsymbol{p}}. \tag{3.17}$$

[4]Note that the analogous relation between the Euclidean group element g and $\boldsymbol{p}$ and $\boldsymbol{j}$ is stated incorrectly at one point in [20]

Note that the formula (3.9) approaches (3.17) in the limit $\mu \to 0$. More generally the relationship between $\boldsymbol{j}$ and $\boldsymbol{k}$ is given by the Lorentz transformation

$$T(\boldsymbol{p}) = \frac{\exp(p^a J_a) - 1}{p^a J_a}. \tag{3.18}$$

Since this transformation, which is the inverse of (3.14), plays an important role in the rest of the paper, we express it more concretely as a power series valid for any value of the momentum three-vector $\boldsymbol{p}$

$$T(\boldsymbol{p}) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(p^a J_a)^n. \tag{3.19}$$

In the case at hand, where $\boldsymbol{p}$ is a time-like vector of length μ, this yields

$$T(\boldsymbol{p}) = \Pi + \frac{\sin\left(\frac{\mu}{2}\right)}{\frac{\mu}{2}} \exp\left(\frac{1}{2}p^a J_a\right)(1 - \Pi), \tag{3.20}$$

where we again used the projector defined after (3.14). Using the adjoint or vector representation of the J_a, i.e. $(J_a)_b{}^c = -\epsilon_{ab}{}^c$, one checks that

$$j_a = T(\boldsymbol{p})_a{}^b k_b. \tag{3.21}$$

Note that in terms of the transformed position vector

$$q_a = T(\boldsymbol{p})_a{}^b x_b, \tag{3.22}$$

the expression for (3.9) takes on the "non-gravitational" form

$$\boldsymbol{j} = \boldsymbol{q} \wedge \boldsymbol{p} + s\hat{\boldsymbol{p}}. \tag{3.23}$$

This formula goes some way towards explaining the non-standard commutation relations of the position coordinates x_a. They differ from those of the q_a (in terms of which j_a has the familiar form) because $\boldsymbol{q}$ and $\boldsymbol{x}$ are related by the $\boldsymbol{p}$-dependent transformation T.

3.4 The Relation to the Dual of the Poisson Lie group $\tilde{P}_3^\uparrow$

For the case of vanishing spin s, our description of the one-particle phase space agrees with the description derived by Matschull and Welling [12], [3] in a completely independent way and generalises their results to the case of arbitrary spin. In order to see this, we have to describe the Poisson structure in terms of a symplectic form rather than a Poisson bivector. It follows from the general results in [7] that the Poisson bivector (3.11) gives the Poisson structure of the dual Poisson-Lie group $(\tilde{P}_3^\uparrow)^*$. In the appendix we show that, as a group, the dual is the direct product $(\tilde{P}_3^\uparrow)^* \cong \tilde{L}_3^\uparrow \times \mathbb{R}^3$. Elements $h = (u, \boldsymbol{a}) \in \tilde{P}_3^\uparrow$ can be factorised uniquely in the form (A.21) $h = h_+ h_-^{-1}$, where $h_+ = (u, 0)$ and $h_- = (1, -\mathrm{Ad}(u^{-1})\boldsymbol{a}) = (1, \boldsymbol{j})$ can naturally be thought of as elements of $(\tilde{P}_3^\uparrow)^*$. The symplectic leaves are precisely the conjugacy classes of h. The symplectic form on those symplectic leaves can be written in terms of the parametrisation (3.5) as [7]

$$\Omega = \frac{1}{2}\langle (dh_+h_+^{-1} - dh_-h_-^{-1}) \wedge dgg^{-1}\rangle. \tag{3.24}$$

Explicitly, we find that $dh_+h_+^{-1}$ is the right-invariant one-form on $\tilde{L}_3^\uparrow$

$$dh_+h_+^{-1} = duu^{-1} \tag{3.25}$$

with values in the Lie algebra $so(2,1)$, and

$$dh_-h_-^{-1} = dj_aP^a = dx_a\left(1 - \mathrm{Ad}(u^{-1})\right)P^a + \mathrm{Ad}(u^{-1})[duu^{-1}, x_aP^a] + s\; d\hat{p}_a\, P^a. \tag{3.26}$$

Using

$$dgg^{-1} = dvv^{-1} + dx_aP^a - [dvv^{-1}, x_aP^a] \quad \text{and} \quad duu^{-1} = (1 - \mathrm{Ad}(u))dvv^{-1}, \tag{3.27}$$

one computes

$$\Omega = \langle duu^{-1} \wedge dx_aP^a\rangle - \langle duu^{-1} \wedge duu^{-1}, x_aP^a\rangle - s\langle v^{-1}dv \wedge v^{-1}dv, P_0\rangle, \tag{3.28}$$

which can be written as the exterior derivative of the symplectic potential

$$\theta = -\langle duu^{-1}, x_aP^a\rangle - s\langle dvv^{-1}, \hat{p}_aP^a\rangle. \tag{3.29}$$

This agrees with the symplectic potential derived in [12] in the spin-less case, provided we identify their u with our u^{-1}, and generalises this expression to the case of non vanishing spin. Finally, note that the symplectic potential can be written very compactly as

$$\theta = -\langle dvv^{-1}, j_aP^a\rangle. \tag{3.30}$$

4 The phase space of N massive particles on a genus g surface with boundary

4.1 Boundary conditions and invariance transformations

After describing the phase space of an open universe with a single particle, we extend our description to a general spacetime manifold $M \approx \mathbb{R} \times S^\infty_{g,n}$ with g handles and n particles of masses $\mu_i \in (0, 2\pi)$ and spins s_i. As we did for the single particle universe, we must first impose an appropriate boundary condition on the gauge fields at spatial infinity and require the corresponding asymptotic behaviour of the gauge transformations. Again, our choice of the boundary condition is modelled after the boundary condition in the metric formalism. In a reference frame where the centre of mass of the universe is at rest at the origin, the physically sensible boundary condition at spatial infinity is the requirement that the metric be asymptotically conical of the form (3.1) [3], i. e. that the universe asymptotically appears like a single particle. Because of the invariance of the conical metric (3.1) under rotations and time translations, the centre of mass condition does not uniquely determine a reference frame. However, we can imagine using a distant particle to fix the orientation and time origin of a distinguished centre of mass frame. This is analogous to the use of distant fixed stars for selecting a reference frame in our (3+1) dimensional universe.

As explained in the single particle discussion in Sect. 3, we do not need to restrict attention to centre of mass frames in the Chern-Simons formulation. Rather we impose the boundary condition that the gauge field at spatial infinity be related to a connection of the form (3.4) by a gauge transformation given asymptotically by a constant element of the Poincaré group, which physically represents Poincaré transformations of the observer relative to the distinguished centre of mass frame. In a universe with several particles and possibly handles, the parameters μ and s in (3.3) stand for the total mass and spin of the universe. Note that we do not impose a fixed value for μ and s at this stage. However, we will find in Sect. 5.2 that μ and s remain constant during the time evolution of the universe. Our boundary condition can be rephrased by saying that the holonomy around the boundary representing infinity is in an elliptic conjugacy class. The equations of motion guarantee that the class remains fixed during the time evolution. Our strategy in the following discussion of the phase space will be to work in a centre of mass frame and then generalise our results to a general frame.

The starting point for our definition of the phase space of $\tilde{P}_3^\uparrow$-Chern-Simons theory is the space of solutions of the equations of motion that satisfy the boundary condition. In a centre of mass frame this is the space $\mathcal{A}_\infty$ of flat connections on $S_{g,n}^\infty$ which are of the form A_∞ (3.4) for some value of μ and s in an open neighbourhood of the boundary. The physical phase space is obtained as a quotient of this space by identifying solutions which represent the same physical state. This means that we have to determine for each invariance of the theory, i. e. for each bijection of the space $\mathcal{A}_\infty$ to itself, if it has an interpretation as a gauge transformation to be divided out of the phase space or gives rise to a physically meaningful symmetry transformation between different states.

The first type of invariance transformation in $\tilde{P}_3^\uparrow$-Chern-Simons theory are gauge transformations (2.20) which are compatible with the boundary conditions at spatial infinity. In a centre of mass frame they are given by the group $\mathcal{P}_\infty$ of $\tilde{P}_3^\uparrow$-valued functions on $S_{g,n}^\infty$ which are a constant rotation or time-translation in a neighbourhood of the boundary and act on the gauge field according to (2.20). It follows from the fact that $\tilde{P}_3^\uparrow$ is contractible (and in particular, simply connected) that all of these transformations are small, i. e. connected to the identity transformation and obtained by exponentiating infinitesimal transformations. Asymptotically trivial transformations which are the identity in a neighbourhood of the boundary form a subgroup $\mathcal{P} \subset \mathcal{P}_\infty$. They are generated by a gauge constraint and transform different descriptions of the same physical state into each other. As redundant transformations without physical meaning, they have to be divided out of the phase space, and field configurations related by them have to be identified. The situation is different for asymptotically nontrivial Chern-Simons gauge transformations. They are not generated directly by a gauge constraint but would require an additional boundary term in the action [22], [13]. They do not correspond to gauge degrees of freedom but give rise to symmetries acting on the phase space and have a physical interpretation of

Poincaré transformations with respect to the distinguished centre of mass frame.

The second type of invariance present in $\tilde{P}_3^{\uparrow}$-Chern-Simons theory arises from the fact that it is a topological theory. It is invariant under the group of all orientation preserving diffeomorphisms compatible with the boundary conditions at spatial infinity. This is the group $\mathcal{D}_\infty$ of orientation preserving diffeomorphisms that reduce to a global spatial rotation in an open neighbourhood of the boundary. Its elements act on the space $\mathcal{A}_\infty$ via pull back with the inverse

$$A \in \mathcal{A}_\infty \to (\phi^{-1})^* A =: \phi_* A \qquad\qquad \phi \in \mathcal{D}_\infty\,. \tag{4.1}$$

Among the elements of $\mathcal{D}_\infty$ we further distinguish the subgroup $\mathcal{D} \subset \mathcal{D}_\infty$ of asymptotically trivial diffeomorphisms, i. e. diffeomorphisms of $S^\infty_{g,n}$ which keep the boundary fixed pointwise. Diffeomorphisms in $\mathcal{D}$ which are connected to the identity are called small diffeomorphisms and form a normal subgroup $\mathcal{D} \subset \mathcal{D}_0$. In a Chern-Simons theory, diffeomorphisms are not a priori related to gauge transformations and there is no reason to consider them as redundant transformations between field configurations describing the same physical state. However, Witten showed in [5] that infinitesimal diffeomorphisms can on-shell be written as infinitesimal Chern-Simons gauge transformations as follows. The infinitesimal transformation of a flat connection A generated by a vector field V is given by the Lie derivative

$$\delta A = \mathcal{L}_V A\,. \tag{4.2}$$

Using the formula $\mathcal{L}_V = di_V + i_V d$ and the flatness of A, this can be written as an infinitesimal gauge transformation

$$\mathcal{L}_V A = d\lambda + [A, \lambda] \tag{4.3}$$

with generator

$$\lambda = i_V A, \tag{4.4}$$

where i_V denotes the contraction with V. By exponentiating, we see that any two field configurations A and A' related by a small, asymptotically trivial diffeomorphism are also related by an asymptotically trivial Chern-Simons gauge transformation and therefore have to be identified. Small diffeomorphisms that reduce to global rotations at the boundary correspond to asymptotically nontrivial Chern-Simons gauge transformations and therefore represent physical transformations with respect to the centre of mass frame. However, the correspondence between diffeomorphisms and Chern-Simons gauge transformations does not hold for large diffeomorphisms, i. e. diffeomorphisms that are not infinitesimally generated. Those diffeomorphisms are not related to gauge degrees of freedom but are physically meaningful transformations acting on the phase space.

We conclude that the physical phase space of $\tilde{P}_3^{\uparrow}$-Chern-Simons theory is obtained from the space $\mathcal{A}_\infty$ of flat gauge connections that satisfy the boundary conditions by identifying connections $A,\ A' \in \mathcal{A}_\infty$ if and only if they are related by an asymptotically trivial Chern-Simons gauge transformation. This implies division by small, asymptotically trivial diffeomorphisms, whereas large diffeomorphisms and asymptotically nontrivial Chern-Simons transformations give rise to physical symmetries. We will return to this question and give a more mathematical treatment in Sect. 5.1, where we determine the symmetry group of the theory.

A final general comment we should make concerns the dreibein e_a in the decomposition of a connection A according to (2.13). In Einstein's metric formulation of gravity the dreibein is assumed to be invertible, but there is no such requirement in Chern-Simons theory. In fact, as pointed out and discussed in [23], gauge orbits under $\tilde{P}_3^\uparrow$ gauge transformations may pass through a connection with a degenerate dreibein. In [23] it is argued that this leads to the identification of spacetimes in the Chern-Simons formulation of (2+1) gravity which would be regarded as physically distinct in the metric formulation. We will address this issue in the final section of this paper, after we have given a detailed description of the phase space of $\tilde{P}_3^\uparrow$-Chern-Simons theory.

4.2 The description of the space time by a graph

The description of the phase space in the formalism of Fock and Rosly depends on a graph. As in the case of a single massive particle, the graphs used in the construction of the Poisson structure on the moduli space have a intuitive physical interpretation. For a (sufficiently refined) graph that is embedded into the surface $S_{g,n}^\infty$ in such a way that the punctures representing massive particles lie on different faces of the graph and at least one vertex is mapped to the boundary at spatial infinity, the vertices of the graph can be thought of as observers located at different spacetime points. As the spacetime manifold is locally flat, the reference frames of these observers should be inertial frames related by Poincaré transformations. These Poincaré transformations are given by the graph connection. The $\tilde{P}_3^\uparrow$-element $g = (v, \mathbf{y})$ assigned to an (oriented) edge can be interpreted as the Poincaré transformation relating the reference frames of the observers at its ends: the translation vector $\mathbf{y}$ describes the shift in position and time coordinate, the Lorentz transformation v the relative orientation of their coordinate axes and their relative velocity. The observers located at different points in the spacetime can determine topology and matter content of their universe by exchanging information about these Poincaré transformations relating their reference frames. Due to the flatness of the graph connection, observers situated at vertices surrounding a face without particles will find that the (ordered) product of the Poincaré transformations along the boundary of the face is equal to the identity. If the face contains a massive particle, the same procedure yields an element of the $\tilde{P}_3^\uparrow$-conjugacy class determined by mass and spin of the particle, thus allowing a measurement of these quantities.

The interpretation of graph connections as Poincaré transformations between reference frames of observers situated at the vertices is complemented by the physical interpretation of graph gauge transformations (2.21). A graph gauge transformation involves an individual $\tilde{P}_3^\uparrow$ transformation at each vertex. As the Poincaré transformations relating adjacent observers have the right transformation property (2.21), the graph gauge transformations can be interpreted as changes of the reference frame for each observer. In this interpretation it is obvious that all transformations not affecting the vertices at the boundary correspond to gauge degrees of freedom. They do not alter the physical state of the universe but only its description in terms of reference frames of different observers. The situation is different, however, for a vertex situated at the boundary. With the interpretation of such a vertex as an observer, the graph gauge transformations affecting the vertex are seen to (uniquely) characterise the asymptotic symmetries discussed above. They correspond to rotations, boosts and translations of the observer with respect to the distinguished centre of mass frame of the universe.

4.3 The phase space of n massive particles and g handles

Taking into account the special role of the boundary, the most efficient Fock-Rosly graph for a surface $S^\infty_{g,n}$ is a set of curves representing the generators of the fundamental group with basepoint z_∞ on the boundary as pictured in Fig. 3.

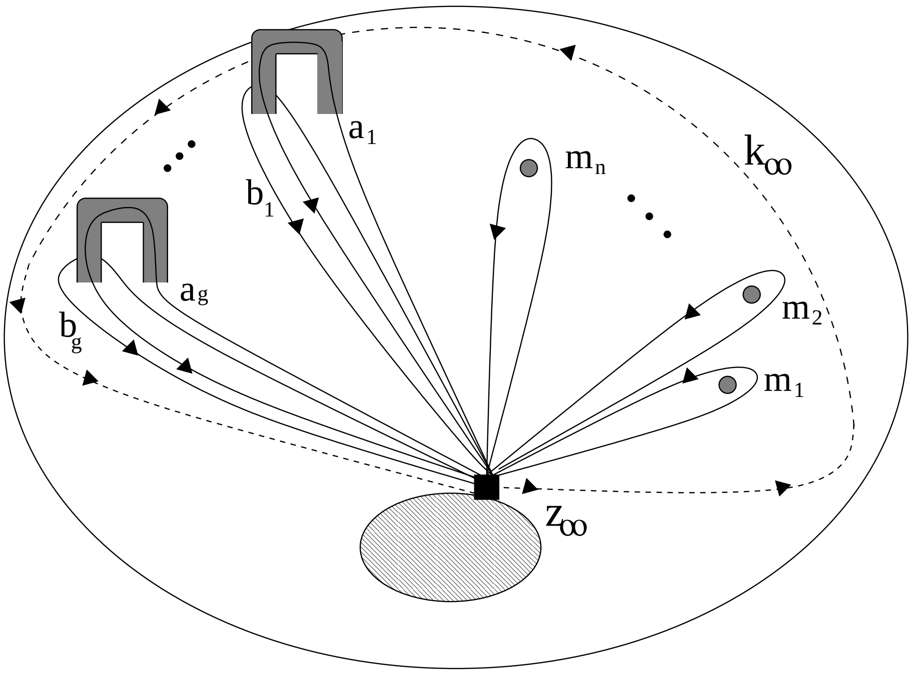

Fig. 3

The generators of the fundamental group of the surface $S^\infty_{g,n}$

The fundamental group is generated by the equivalence classes of the curves $m_1, \ldots, m_n$ around each particle, curves $a_1, b_1, \ldots, a_g, b_g$ around each handle and a curve around the boundary at spatial infinity, subject to a relation similar to (2.26). By solving the relation, the fundamental group can be presented as the free group generated by the curves around the particles and handles

$$\pi_1(z_\infty, S^\infty_{g,n}) =< m_1, \ldots, m_n, a_1, b_1, \ldots, a_g, b_g > . \tag{4.5}$$

The holonomy along these curves assigns elements $M_i \in \tilde{P}^\uparrow_3$, $i = 1, \ldots, n$, A_j, $B_j \in \tilde{P}^\uparrow_3$, $j = 1, \ldots, g$ of the gauge group $\tilde{P}^\uparrow_3$ to each of the generators m_i, a_j, b_j. Asymptotically nontrivial Chern-Simons gauge transformations (2.20) induce graph gauge transformations that act on these group elements via simultaneous conjugation. As explained in the previous subsection, these are physical symmetries and not divided out of the physical phase space.

The same is true for the group of asymptotically trivial, large diffeomorphisms. Mathematically, this is called the mapping class group and defined as

$$\mathrm{Map}(S^\infty_{g,n}) = \mathcal{D}/\mathcal{D}_0. \tag{4.6}$$

In our description of the spacetime by means of an embedded graph, asymptotically trivial, large diffeomorphisms map the graph to a different, topologically inequivalent graph. If we include the large diffeomorphisms as symmetries acting on the phase space, our phase space is not simply given as the quotient of the space of graph connections modulo (asymptotically trivial) graph gauge transformations corresponding to a *fixed graph*. Rather, we consider topologically distinct graphs and include an additional label specifying the graph. In the case where the graph is given by a set of generators of the fundamental group, this can be made more explicit.

The starting point for our definition of the phase space is one fixed graph which consists of the generators of the fundamental group $\pi_1(z_\infty, S^\infty_{g,n})$ shown in Fig. 3. With respect to this graph, the phase space is simply the product

$$\mathcal{C}_1 \times \cdots \times \mathcal{C}_n \times (\tilde{P}_3^\uparrow)^{2g}\;, \tag{4.7}$$

where

$$\mathcal{C}_i = \{g\,(\exp(-\mu_i J^0), -(s_i,0,0)^t)\, g^{-1} \mid g \in \tilde{P}_3^\uparrow\} \qquad i = 1,\ldots,n \tag{4.8}$$

are the $\tilde{P}_3^\uparrow$-conjugacy classes determined by masses and spins of the particles.

The mapping class group $\mathrm{Map}(S^\infty_{g,n})$ acts on $\pi_1(z_\infty, S^\infty_{g,n})$ and induces a subgroup of the automorphism group of $\pi_1(z_\infty, S^\infty_{g,n})$ [24],[25]. The graphs related to the one depicted in Fig. 3 by an element of $\mathrm{Map}(S^\infty_{g,n})$ are therefore conveniently labelled by that element. Finally, by associating to every element of the mapping class group the induced permutation of the punctures $z_1, \ldots z_n$, we get a homomorphism into the permutation group P_n

$$\pi : \mathrm{Map}(S^\infty_{g,n}) \to P_n, \tag{4.9}$$

in terms of which the phase space is defined as

$$\begin{aligned}\mathcal{M}_{\pi_1} \;=\; &\{(\sigma, (M_1, \ldots, M_n, A_1, B_1, \ldots, A_g, B_g)) \in \mathrm{Map}(S^\infty_{g,n}) \times (\tilde{P}_3^\uparrow)^{n+2g}\\ &\mid M_i \in \mathcal{C}_{\pi(\sigma)(i)},\; i = 1, \ldots, n\}.\end{aligned} \tag{4.10}$$

We use the following parametrisation of the phase space. As the holonomies of the loops around the punctures are elements of fixed, elliptic $P_3^\uparrow$-conjugacy classes and the particle's masses are in the interval $(0, 2\pi)$, they can be parametrised by

$$M_i = (u_{M_i}, \boldsymbol{a}^{M_i}) \qquad \text{with} \qquad \begin{aligned}&u_{M_i} = \exp(-p^a_{M_i} J_a),\; \boldsymbol{a}^{M_i} = -\mathrm{Ad}(u_{M_i})\boldsymbol{j}^{M_i},\\ &p^0_{M_i} > 0,\; \boldsymbol{p}^2_{M_i} = \mu_i^2,\; \mu_i \in (0, 2\pi).\end{aligned} \tag{4.11}$$

In terms of the parameters $p_a^{M_i}$, $j_a^{M_i}$ the constraints then take the form (2.6)

$$p_a^{M_i} p^a_{M_i} \approx \mu_i^2 \qquad p_a^{M_i} j^a_{M_i} \approx \mu_i s_i \qquad i = 1, \ldots, n, \tag{4.12}$$

in agreement with the interpretation of $\mu_i \in (0, 2\pi)$ and s_i as masses and spins of the particles. There are no such constraints for the holonomies of the loops around the handles. They can be elliptic, parabolic or hyperbolic elements of $\tilde{P}_3^\uparrow$ and therefore have to be parametrised in the more general form (2.4)

$$\begin{aligned} A_i &= (u_{A_i}, \boldsymbol{a}_{A_i}) \quad \text{with } u_{A_i} = \exp(-2\pi n_{A_i} J_0) \exp(-p^a_{A_i} J_a),\ \boldsymbol{a}_{A_i} = -\mathrm{Ad}(u_{A_i}) \boldsymbol{j}^{A_i} \\ B_i &= (u_{B_i}, \boldsymbol{a}_{B_i}) \quad \text{with } u_{B_i} = \exp(-2\pi n_{B_i} J_0) \exp(-p^a_{B_i} J_a),\ \boldsymbol{a}_{B_i} = -\mathrm{Ad}(u_{B_i}) \boldsymbol{j}^{B_i}, \end{aligned} \tag{4.13}$$

where $n_{A_i}, n_{B_i} \in \mathbb{Z}$.

4.4 Poisson structure

As the graph given by the fundamental group has only one vertex, the Poisson structure on the phase space involves the choice of a single r-matrix. It is given by the bivector introduced in [11]

$$\begin{aligned} B_{FR} =& \sum_{i=1}^n r^{\alpha\beta} \left(\frac{1}{2} R^{M_i}_\alpha \wedge R^{M_i}_\beta + \frac{1}{2} L^{M_i}_\alpha \wedge L^{M_i}_\beta + R^{M_i}_\alpha \wedge L^{M_i}_\beta \right) \\ &+ \sum_{i=1}^g r^{\alpha\beta} \Big(\frac{1}{2} \left(R^{A_i}_\alpha \wedge R^{A_i}_\beta + L^{A_i}_\alpha \wedge L^{A_i}_\beta + R^{B_i}_\alpha \wedge R^{B_i}_\beta + L^{B_i}_\alpha \wedge L^{B_i}_\beta \right) \\ &\qquad + R^{A_i}_\alpha \wedge (R^{B_i}_\beta + L^{A_i}_\beta + L^{B_i}_\beta) + R^{B_i}_\alpha \wedge (L^{A_i}_\beta + L^{B_i}_\beta) + L^{A_i}_\alpha \wedge L^{B_i}_\beta \Big) \\ &+ \sum_{1 \leq i < j \leq n} r^{\alpha\beta} (R^{M_i}_\alpha + L^{M_i}_\alpha) \wedge (R^{M_j}_\beta + L^{M_j}_\beta) \\ &+ \sum_{1 \leq i < j \leq g} r^{\alpha\beta} (R^{A_i}_\alpha + L^{A_i}_\alpha + R^{B_i}_\alpha + L^{B_i}_\alpha) \wedge (R^{A_j}_\beta + L^{A_j}_\beta + R^{B_j}_\beta + L^{B_j}_\beta) \\ &+ \sum_{i=1}^n \sum_{j=1}^g r^{\alpha\beta} (R^{M_i}_\alpha + L^{M_i}_\alpha) \wedge (R^{A_j}_\beta + L^{A_j}_\beta + R^{B_j}_\beta + L^{B_j}_\beta), \end{aligned} \tag{4.14}$$

where $M_1, \ldots, M_n$ denote the holonomies around the particles, $A_1, B_1, \ldots, A_g, B_g$ the holonomies corresponding to the handles and the cilium at the vertex z_∞ is chosen such that the order of the incident edges is

$$m_1 < \ldots < m_n < a_1 < b_1 < a_1^{-1} < b_1^{-1} < \ldots < a_g < b_g < a_g^{-1} < b_g^{-1}.$$

As there is no division by global conjugation, the correct r-matrix has again to be determined from a *physical* requirement, namely, that the Poisson brackets of total momentum $\boldsymbol{p}_{tot}$ and total angular momentum $\boldsymbol{j}_{tot}$ of the universe are those of a free particle and act as the infinitesimal generators of asymptotic Poincaré transformations. As in the case of a single particle, this requires the r-matrix $r = P_a \otimes J^a$. Taking into account that the elements $\exp(-2\pi n J_0)$ with $n \in \mathbb{Z}$ are central in $\tilde{P}_3^\uparrow$, we obtain expressions of the form (3.13) for the right and left invariant vector fields corresponding to the generators of the fundamental group. Inserting these fields and the r-matrix into equation (4.14), we derive the Poisson structure:

$$
\begin{aligned}
&\{p^a_{M_i}, p^b_{M_i}\} = 0 && i = 1, \ldots, n && (4.15)\\
&\{j^a_{M_i}, p^b_{M_i}\} = -\epsilon^{abc} p_c^{M_i} \\
&\{j^a_{M_i}, j^b_{M_i}\} = -\epsilon^{abc} j_c^{M_i}
\end{aligned}
$$

$$
\begin{aligned}
&\{j^a_X, p^b_Y\} = -(1 - \mathrm{Ad}(u_X))_d{}^a \epsilon^{bcd} p_c^Y && X < Y, && (4.16)\\
&\{j^a_X, j^b_Y\} = -(1 - \mathrm{Ad}(u_X))_d{}^a \epsilon^{bcd} j_c^Y && X, Y \in \{M_1, \ldots, M_n, A_1, \ldots, B_g\} \\
&\{p^a_X, j^b_Y\} = 0 \\
&\{p^a_X, p^b_Y\} = 0
\end{aligned}
$$

$$
\begin{aligned}
&\{p^a_{A_i}, p^b_{A_i}\} = 0 && i = 1, \ldots, g \qquad (4.17)\\
&\{j^a_{A_i}, p^b_{A_i}\} = -\epsilon^{abc} p_c^{A_i} \\
&\{j^a_{A_i}, j^b_{A_i}\} = -\epsilon^{abc} j_c^{A_i} \\
&\{p^a_{B_i}, p^b_{B_i}\} = 0 \\
&\{j^a_{B_i}, p^b_{B_i}\} = -\epsilon^{abc} p_c^{B_i} \\
&\{j^a_{B_i}, j^b_{B_i}\} = -\epsilon^{abc} j_c^{B_i} \\
&\{j^a_{A_i}, j^b_{B_i}\} = -\epsilon^{abc} j_c^{B_i} \\
&\{p_{A_i}, p_{B_i}\} = 0 \\
&\{j^a_{A_i}, p^b_{B_i}\} = -\left(1 + \mathrm{Ad}(u_{A_i}^{-1}) \cdot \frac{\mathrm{Ad}(u_{B_i})}{1 - \mathrm{Ad}(u_{B_i})}\right)_d^{\ a} \epsilon^{bcd} p_c^{B_i} = \quad -\epsilon^{abc} p_c^{B_i} + \mathrm{Ad}(u_{A_i}^{-1})^a{}_c (T^{-1}(\boldsymbol{p}_{B_i}))^{cb} \\
&\{j^a_{B_i}, p^b_{A_i}\} = \left(\frac{\mathrm{Ad}(u_{A_i})}{1 - \mathrm{Ad}(u_{A_i})}\right)_d^{\ a} \epsilon^{bcd} p_c^{A_i} = -(T^{-1}(\boldsymbol{p}_{A_i}))^{ab},
\end{aligned}
$$

where $M_1 < \cdots < M_n < A_1, B_1 < \cdots < A_g, B_g$. The maps $T(\boldsymbol{p}_{A_i}), T(\boldsymbol{p}_{B_i})$ are given by (3.18) and their inverses by (3.14). Note that the Poisson structure is not the direct product of the Poisson structures of different particles and handles but mixes their contributions. Furthermore, it can be seen from equations (4.15) to (4.17) that the mass and spin constraints (4.12) for the particles Poisson commute with all functions on the phase space. This is not true for the corresponding expressions for the handles, as we have from (4.17)

$$\{\boldsymbol{p}_{B_i} \cdot \boldsymbol{j}_{B_i}, \boldsymbol{p}_{A_i}^2\} = -\{\boldsymbol{p}_{A_i} \cdot \boldsymbol{j}_{A_i}, \boldsymbol{p}_{B_i}^2\} \;=\; 2\boldsymbol{p}_{A_i} \cdot \boldsymbol{p}_{B_i}. \qquad (4.18)$$

In fact, we will see at the end of this section that the Poisson structure corresponding to a single handle has no Casimir functions and is symplectic.

4.5 The decoupling transformation

In [8], Alekseev and Malkin discovered a bijective transformation of the moduli space of flat G connections that maps its symplectic structure onto the direct product of n copies of the symplectic form on the conjugacy classes of G and g copies of the symplectic form on the Heisenberg double $\mathcal{D}_+(G)$ as defined in the appendix. They proved their result for the case of a compact semi-simple Lie group G and an underlying closed surface with punctures. For our purposes we need an extension of their result to non-compact gauge groups and a surfaces with boundary. In order to state our result as simply as possible we define the inverse of the transformation in [8]. With the factorisation (A.21) $(u,\boldsymbol{a}) = (u,\boldsymbol{a})_+(u,\boldsymbol{a})_-^{-1}$ discussed in the appendix, it is given by

$$K^{-1}: \mathcal{C}_1\times\cdots\times\mathcal{C}_n\times\underbrace{\mathcal{D}_+(\tilde{P}_3^\uparrow)\times\cdots\times\mathcal{D}_+(\tilde{P}_3^\uparrow)}_{g\times}\longrightarrow \mathcal{C}_1\times\cdots\times\mathcal{C}_n\times\underbrace{\tilde{P}_3^\uparrow\times\cdots\times\tilde{P}_3^\uparrow}_{2g\times}$$

$$\begin{aligned}
M_i' = (u_{M_i'},\boldsymbol{a}_{M_i'}) &\quad\longrightarrow\quad M_i = (u_{M_i},\boldsymbol{a}_{M_i}) \\
&\qquad\qquad = M'_{(n+2g)-}\cdots M'_{(i+1)-}\cdot M_i'\cdot M'^{-1}_{(i+1)-}\cdots M'^{-1}_{(n+2g)-} \\
A_i' = (u_{A_i'},\boldsymbol{a}_{A_i'}) &\quad\longrightarrow\quad A_i = (u_{A_i},\boldsymbol{a}_{A_i}) \\
&\qquad\qquad = M'_{(n+2g)-}\ldots M'_{(n+2i+1)-}\cdot A_i'\cdot M'^{-1}_{(n+2i)-}\cdots M'^{-1}_{(n+2g)-} \\
B_i' = (u_{B_i'},\boldsymbol{a}_{B_i'}) &\quad\longrightarrow\quad B_i = (u_{B_i},\boldsymbol{a}_{B_i}) \\
&\qquad\qquad = M'_{(n+2g)-}\cdots M'_{(n+2i+1)-}\cdot B_i'\cdot M'^{-1}_{(n+2i)-}\cdots M'^{-1}_{(n+2g)-}
\end{aligned}\tag{4.19}$$

with

$$M'_{(n+2i)} := B_i'A_i'^{-1}\qquad\text{and}\qquad M'_{(n+2i-1)} := B_i'^{-1}A_i'\,.$$

Then our result can be stated as follows.

Theorem 4.1 *1. The transformation K^{-1} given by (4.19) transforms the momentum and angular momentum three-vectors of particles and handles according to*

$$\begin{aligned}
&\boldsymbol{p}^{M_i} = \boldsymbol{p}^{M_i'},\qquad \boldsymbol{p}^{A_i} = \boldsymbol{p}^{A_i'},\qquad \boldsymbol{p}^{B_i} = \boldsymbol{p}^{B_i'} \\
&u_{M_i} = u_{M_i'},\qquad u_{A_i} = u_{A_i'},\qquad u_{B_i} = u_{B_i'} \\
&\boldsymbol{j}^{M_i} = \boldsymbol{j}^{M_i'} + (1-\mathrm{Ad}(u^{-1}_{M_i'}))\left(\sum_{j=i+1}^{n}\boldsymbol{j}^{M_j'}\right) \\
&\qquad + (1-\mathrm{Ad}(u^{-1}_{M_i'}))\left(\sum_{j=1}^{g}\Big(1-\mathrm{Ad}(u_{A_j'})\Big)\,\boldsymbol{j}^{A_j'} + \Big(\mathrm{Ad}(u_{A_j'})-\mathrm{Ad}(u^{-1}_{A_j'}u_{B_j'})\Big)\,\boldsymbol{j}^{B_j'}\right) \\
&\boldsymbol{j}^{A_i} = \boldsymbol{j}^{A_i'} - \mathrm{Ad}(u_{A_i'})(\boldsymbol{j}^{A_i'}-\boldsymbol{j}^{B_i'}) \\
&\qquad + (1-\mathrm{Ad}(u^{-1}_{A_i'}))\left(\sum_{j=i+1}^{g}\Big(1-\mathrm{Ad}(u_{A_j'})\Big)\,\boldsymbol{j}^{A_j'} + \Big(\mathrm{Ad}(u_{A_j'})-\mathrm{Ad}(u^{-1}_{A_j'}u_{B_j'})\Big)\,\boldsymbol{j}^{B_j'}\right) \\
&\boldsymbol{j}^{B_i} = \boldsymbol{j}^{B_i'} - \mathrm{Ad}(u_{A_i'})(\boldsymbol{j}^{A_i'}-\boldsymbol{j}^{B_i'}) \\
&\qquad + (1-\mathrm{Ad}(u^{-1}_{B_i'}))\left(\sum_{j=i+1}^{g}\Big(1-\mathrm{Ad}(u_{A_j'})\Big)\,\boldsymbol{j}^{A_j'} + \Big(\mathrm{Ad}(u_{A_j'})-\mathrm{Ad}(u^{-1}_{A_j'}u_{B_j'})\Big)\,\boldsymbol{j}^{B_j'}\right).
\end{aligned}\tag{4.20}$$

2. *Its inverse K maps the Poisson structure given by (4.15) to (4.17) to the direct product of n symplectic forms on the conjugacy classes and g Poisson structures on the Heisenberg double $\mathcal{D}_+(g)$*

$$\begin{aligned}
&\{p^a_{M'_i}, p^b_{M'_i}\} = 0 \qquad\qquad\qquad\qquad i = 1,\ldots,n \\
&\{j^a_{M'_i}, p^b_{M'_i}\} = -\epsilon^{abc} p_c^{M'_i} \\
&\{j^a_{M'_i}, j^b_{M'_i}\} = -\epsilon^{abc} j_c^{M'_i}
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
&\{j^a_X, p^b_Y\} = 0 \qquad X < Y,\ X, Y \in \{M'_1,\ldots,M'_n, A'_1,\ldots,B'_g\} \\
&\{j^a_X, j^b_Y\} = 0 \\
&\{p^a_X, j^b_Y\} = 0 \\
&\{p^a_X, p^b_Y\} = 0
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
&\{p^a_{A'_i}, p^b_{A'_i}\} = 0 \qquad\qquad\qquad\qquad i = 1,\ldots,g \\
&\{j^a_{A'_i}, p^b_{A'_i}\} = -\left(\frac{1}{1-\mathrm{Ad}(u_{A'_i})}\right)^a_{\ d} \epsilon^{bcd} p_c^{A'_i} = \mathrm{Ad}(u^{-1}_{A'_i})^a{}_c (T^{-1}(\boldsymbol{p}_{A'_i}))^{cb} \\
&\{j^a_{A'_i}, j^b_{A'_i}\} = -\epsilon_{abc} j_c^{A'_i} \\
&\{p^a_{B'_i}, p^b_{B'_i}\} = 0 \\
&\{j^a_{B'_i}, p^b_{B'_i}\} = -\left(\frac{1}{1-\mathrm{Ad}(u_{B'_i})}\right)^a_{\ d} \epsilon^{bcd} p_c^{B'_i} = \mathrm{Ad}(u^{-1}_{B'_i})^a{}_c (T^{-1}(\boldsymbol{p}_{B'_i}))^{cb} \\
&\{j^a_{B'_i}, j^b_{B'_i}\} = -\epsilon_{abc} j_c^{B'_i} \\
&\{p^a_{A'_i}, p^b_{B'_i}\} = 0 \\
&\{j^a_{A'_i}, p^b_{B'_i}\} = -\left(\left(1+\mathrm{Ad}(u^{-1}_{A'_i} u_{B'_i})\right)\cdot\frac{1}{1-\mathrm{Ad}(u_{B'_i})}\right)^a_{\ d} \epsilon^{bcd} p_c^{B'_i} \\
&\qquad\qquad\quad = \left(\mathrm{Ad}(u^{-1}_{B'_i}) + \mathrm{Ad}(u^{-1}_{A'_i})\right)^a_{\ c} (T^{-1}(\boldsymbol{p}_{B'_i}))^{cb} \\
&\{j^a_{A'_i}, j^b_{B'_i}\} = -\epsilon^{abc} j_c^{B'_i} \\
&\{j^a_{B'_i}, p^b_{A'_i}\} = 0
\end{aligned} \tag{4.23}$$

where, again, $M'_1 < \cdots < M'_n < A'_1, B'_1 < \cdots < A'_g, B'_g$, and the maps $T(\boldsymbol{p}_{A'_i}), T(\boldsymbol{p}_{B'_i})$ are given by (3.18), their inverses by (3.14).

Proof: The result follows by a direct but lengthy computation. The following steps and formulae are essential.

1. We obtain the transformation of the parameters $\boldsymbol{p}, \boldsymbol{j}$ by inserting $(u, \boldsymbol{a})_- = (1, -\mathrm{Ad}(u^{-1})\boldsymbol{a}) = (1, \boldsymbol{j})$ in (4.19), which yields (4.20) by straightforward calculation. Further calculation proves that K^{-1} is indeed bijective and can be inverted.

2. We show that the transformation K^{-1} given by equation (4.20) transforms the brackets (4.21) to (4.23) into the Poisson brackets (4.15) to (4.17). Defining

$$\begin{aligned}
\boldsymbol{v}^{A_i'} &= \boldsymbol{j}^{A_i'} - \mathrm{Ad}(u_{A_i})(j^{A_i'} - j^{B_i'}) \\
\boldsymbol{v}^{B_i'} &= \boldsymbol{j}^{B_i'} - \mathrm{Ad}(u_{A_i})(j^{A_i'} - j^{B_i'}) \\
\boldsymbol{j}^{H_i'} &= (1 - \mathrm{Ad}(u_{A_i}))\,\boldsymbol{j}^{A_i'} + \left(\mathrm{Ad}(u_{A_i}) - \mathrm{Ad}(u_{A_i}^{-1}u_{B_i})\right)\boldsymbol{j}^{B_i'}
\end{aligned} \tag{4.24}$$

for $i = 1, \ldots, g$, equations (4.20) read

$$\begin{aligned}
\boldsymbol{j}^{M_i} &= \boldsymbol{j}^{M_i'} + (1 - \mathrm{Ad}(u_{M_i'}^{-1}))\left(\sum_{j=i+1}^{n}\boldsymbol{j}^{M_j'} + \sum_{j=1}^{g}\boldsymbol{j}^{H_j'}\right) \qquad & i = 1, \ldots, n \\
\boldsymbol{j}^{A_i} &= \boldsymbol{v}^{A_i'} + (1 - \mathrm{Ad}(u_{A_i'}^{-1}))\left(\sum_{j=i+1}^{g}\boldsymbol{j}^{H_j'}\right) & i = 1, \ldots, g \\
\boldsymbol{j}^{B_i} &= \boldsymbol{v}^{B_i'} + (1 - \mathrm{Ad}(u_{B_i'}^{-1}))\left(\sum_{j=i+1}^{g}\boldsymbol{j}^{H_j'}\right) & i = 1, \ldots, g.
\end{aligned} \tag{4.25}$$

We insert these expressions into (4.21) to (4.23) and calculate the resulting Poisson brackets, using the expression (2.5) for the adjoint in terms of $\boldsymbol{p}$, the Ad-invariance of the epsilon-tensor as well as the following auxiliary formulas

$$\begin{aligned}
&\{v_a^{A_i'}, \mathrm{Ad}(u_{A_i'})_{bc}(v_c^{A_i'} - v_c^{B_i'})\} = \{v_a^{B_i'}, \mathrm{Ad}(u_{A_i'})_{bc}(v_c^{A_i'} - v_c^{B_i'})\} = -\epsilon_{abc}\mathrm{Ad}(u_{A_i'})^{ce}(v_e^{A_i'} - v_e^{B_i'}) \\
&\{j_a^{H_i'}, j_b^{H_i'}\} = -\epsilon_{abc} j_{H_i'}^c \\
&\{j_a^{H_i'}, v_b^{X'}\} = -\epsilon_{abc} v_{X'}^c \\
&\{j_a^{H_i'}, p_b^{X'}\} = -\epsilon_{abc} p_{X'}^c \\
&\{\mathrm{Ad}(u_{X'})_{ab}, j_c^{H_i'}\} = -\epsilon_{ace}\mathrm{Ad}(u_{X'})^e{}_b - \epsilon_{bce}\mathrm{Ad}(u_{X'})_a{}^e \qquad \text{for } X' = A_i, B_i.
\end{aligned}$$

After a rather complicated calculation, we obtain brackets (4.15) to (4.17). □

We see that the Poisson brackets (4.21) are just the direct sum of n single particle brackets, i. e. of n copies of the Poisson structure on the $\tilde{P}_3^{\uparrow}$-conjugacy classes. The Poisson brackets (4.22) of three-momenta and the angular momentum three-vectors associated to handles and particles with those associated to different handles vanish. The contribution (4.23) of each handle agrees with the expression (A.25) for the symplectic structure on the Heisenberg double given in the appendix. We shall see in the next section that the total angular momentum three-vector has a particularly simple expression in terms of the decoupled coordinates.

5 Symmetries and the action of the mapping class group

5.1 The symmetry group in the Chern-Simons formulation of (2+1) gravity

As explained in Sect. 4.1, two flat connections in $\mathcal{A}_\infty$ describe the same physical state if and only if they can be transformed into each other by an asymptotically trivial Chern-Simons gauge transformation. Due to the relation (4.3), this is the case for connections related by small, asymptotically trivial diffeomorphisms, whereas large diffeomorphisms and asymptotically nontrivial $\tilde{P}_3^{\uparrow}$- gauge transformations give rise to physical symmetries. However, we have yet to determine exactly the symmetry group of our model and its action on the physical phase space.

The question of the symmetry group of gauge theories or general relativity on a manifold with a boundary was investigated by Giulini in [13],[26]. He classifies the transformations that map the space of solutions to itself into three categories: the group $\mathcal{T}^\infty$ of transformations compatible with the boundary condition, the group $\mathcal{T}$ of transformations that are trivial at the boundary and its identity component, the group $\mathcal{T}_0$ of infinitesimally generated transformations that are trivial at the boundary. Among these, only the transformations in $\mathcal{T}_0$ are generated by the gauge constraints and map solutions to physically equivalent solutions that should be identified. The other two types of transformations do not correspond to gauge degrees of freedom but are related to physical symmetries. The quotient $\mathcal{B} = \mathcal{T}^\infty/\mathcal{T}$ is the asymptotic symmetry group, reflecting the symmetries of the boundary condition, and the symmetry group $\mathcal{S}$ of the theory is given as the quotient $\mathcal{S} = \mathcal{T}^\infty/\mathcal{T}_0$. Giulini showed that these quotients are related by a bundle structure

$$\begin{array}{ccc} \mathcal{T}/\mathcal{T}_0 & \xrightarrow{i} & \mathcal{S} = \mathcal{T}_\infty/\mathcal{T}_0 \\ & & \downarrow p \\ & & \mathcal{B} = \mathcal{T}_\infty/\mathcal{T}. \end{array} \tag{5.1}$$

If the bundle is trivial, the symmetry group is simply the product $\mathcal{S} = \mathcal{T}/\mathcal{T}_0 \times \mathcal{B}$ of the group $\mathcal{T}/\mathcal{T}_0$ and the asymptotic symmetry group. In general, this is not the case and the symmetry group has a more complicated structure.

We now apply the results of [13] and [26] to our model of $\tilde{P}_3^{\uparrow}$-Chern-Simons theory on a surface $S_{g,n}^\infty$, starting with the definition of the groups $\mathcal{T}_\infty$, $\mathcal{T}$ and $\mathcal{T}_0$. The group $\mathcal{T}_\infty$ of transformations compatible with the boundary conditions combines the group $\mathcal{D}_\infty$ of diffeomorphisms that reduce to global rotations and the group $\mathcal{P}_\infty$ of $\tilde{P}_3^{\uparrow}$-transformations which reduce to constant spatial rotations and time-translations at the boundary. The actions of these groups on the space $\mathcal{A}_\infty$ are given by equation (4.1) and (2.20), respectively. Considering their semidirect product $\mathcal{D}_\infty \ltimes \mathcal{P}_\infty$ with group multiplication

$$(\phi_1, \gamma_1)(\phi_2, \gamma_2) = (\phi_1 \circ \phi_2, \gamma_1 \left(\gamma_2 \circ \phi_1^{-1}\right)) \tag{5.2}$$

and combining their actions acoording to

$$A \in \mathcal{A}_\infty \to \gamma \phi_* A \gamma^{-1} + \gamma d\gamma^{-1} \qquad \text{for } \phi \in \mathcal{D}_\infty,\ \gamma \in \mathcal{P}_\infty, \tag{5.3}$$

we obtain a homomorphism

$$\rho_\infty : \mathcal{D}_\infty \ltimes \mathcal{P}_\infty \to \mathrm{Aut}(\mathcal{A}_\infty) \tag{5.4}$$

into the space $\mathrm{Aut}(\mathcal{A}_\infty)$ of bijections of $\mathcal{A}_\infty$ to itself. The group $\mathcal{T}_\infty$ is obtained from the semidirect product $\mathcal{D}_\infty \ltimes \mathcal{P}_\infty$ by identifying diffeomorphisms and Chern-Simons gauge transformations that are identical *as maps* of $\mathcal{A}_\infty$ to itself, i. e. by dividing out the kernel of ρ_∞

$$\mathcal{T}_\infty = (\mathcal{D}_\infty \ltimes \mathcal{P}_\infty)/\ker \rho_\infty\,. \tag{5.5}$$

Note in this context that the equivalence (4.3) between infinitesimal diffeomorphisms and Chern-Simons gauge transformations only holds for a given connection $A \in \mathcal{A}_\infty$ and does not imply that these transformations induce the same maps of $\mathcal{A}_\infty$ via the homomorphism ρ_∞.

The group $\mathcal{T}$ of asymptotically trivial invariance transformations combines asymptotically trivial diffeomorphisms with asymptotically trivial Chern-Simons gauge transformations. We define the homomorphism $\rho : \mathcal{D} \ltimes \mathcal{P} \to \mathrm{Aut}(\mathcal{A}_\infty)$ as the restriction of ρ_∞ (5.4) to $\mathcal{D} \ltimes \mathcal{P}$. Then the group $\mathcal{T}$ is given by

$$\mathcal{T} = (\mathcal{D} \ltimes \mathcal{P})/\ker \rho\,. \tag{5.6}$$

Finally, the group $\mathcal{T}_0$ of small, asymptotically trivial transformations combines the group $\mathcal{D}_0$ of infinitesimally generated, asymptotically trivial diffeomorphisms with asymptotically trivial Chern-Simons transformations. In terms of the restriction ρ_0 of ρ_∞ to $\mathcal{D}_0 \ltimes \mathcal{P}_0$ we have

$$\mathcal{T}_0 = (\mathcal{D}_0 \ltimes \mathcal{P})/\ker \rho_0, \tag{5.7}$$

where we have again used that fact that, due to the contractibility of $\tilde{P}_3^\uparrow$, all elements of $\mathcal{P}$ are connected to the identity.

We are now ready to determine the asymptotic symmetry group $\mathcal{B}$ and the symmetry group $\mathcal{S}$ of $\tilde{P}_3^\uparrow$-Chern-Simons theory as defined in (5.1). Using $\ker \rho = \ker \rho_\infty \cap \mathcal{T}$ and $\ker \rho_0 = \ker \rho_\infty \cap \mathcal{T}_0$ and standard results for quotients involving normal subgroups (see for example the book [27] by Lang), it follows that these groups are given by

$$\mathcal{B} = \mathcal{T}_\infty/\mathcal{T} = \frac{(\mathcal{D}_\infty \ltimes \mathcal{P}_\infty)/(\mathcal{D} \ltimes \mathcal{P})}{\ker \rho_\infty/\ker \rho} \tag{5.8}$$

$$\mathcal{S} = \mathcal{T}_\infty/\mathcal{T}_0 = \frac{(\mathcal{D}_\infty \ltimes \mathcal{P}_\infty)/(\mathcal{D}_0 \ltimes \mathcal{P})}{\ker \rho_\infty/\ker \rho_0}\,. \tag{5.9}$$

In order to evaluate these quotients we have to clarify the relationship between the kernels of the maps ρ_∞, ρ and ρ_0. We claim

Lemma 5.1 *For the maps ρ_∞, ρ and ρ_0 defined in* (5.5),(5.6) *and* (5.7) *we have*

$$\mathit{ker}\,\rho = \mathit{ker}\,\rho_0 \tag{5.10}$$

$$\mathit{ker}\,\rho_\infty/\mathit{ker}\,\rho \cong \tilde{U}(1). \tag{5.11}$$

Proof

1. Clearly, $\ker \rho_0 \subseteq \ker \rho$, but to prove $\ker \rho_0 = \ker \rho$ we need to show that it is impossible for (ϕ, γ) to be in $\ker \rho$ if ϕ is a large diffeomorphism. However, this follows from the explicit determination of the action of the mapping class group on the holonomies we give in Sect. 5.3. Suppose ϕ is a large diffeomorphism and $A \in \mathcal{A}_\infty$. Computing the holonomies of $\phi_* A$ around the generators $m_1, ...m_n, a_1, b_1, ...a_g, b_g$

shown in Fig. 3 is equivalent to computing the holonomies of A around the transformed generators $\phi^{-1} \circ m_1$ etc. However, we show explicitly in Sect. 5.3 that non-trivial elements of the mapping act non-trivially on the holonomies. On the other hand, gauge transformations which are 1 at the boundary act trivially. Thus no such pair (ϕ, γ) can be in ker ρ.

2. To establish (5.11) we show that the evaluation map ker $\rho_\infty \to \tilde{U}(1)$, which assigns to each element $(\phi, \gamma) \in$ ker ρ_∞ the $\tilde{U}(1)$ rotation in $\gamma(\infty)$ is a surjective homomorphism with kernel ker ρ. It is clear that the map is a homomorphism; to see that it is surjective recall the special form of any connection in $\mathcal{A}_\infty$ in a neighbourhood of the boundary. Equations (3.3) and (3.4) imply that any gauge transformation γ_α that reduces to a rotation of angle $-\alpha$ near the boundary can be compensated by a diffeomorphism $\phi_{\{\alpha\}}$ which asymptotically is a global rotation by the mod 2π reduction $\{\alpha\} \in [0, 2\pi)$ of α. Hence the pair $(\phi_{\{\alpha\}}, \gamma_\alpha)$ is in ker ρ_∞ for any $\alpha \in \mathbb{R}$.□

It seems very plausible that in fact ker $\rho =$ ker $\rho_0 \cong 1$ and ker $\rho_\infty \cong \tilde{U}(1)$. We have not been able to show this rigorously, but for the following discussion the relations (5.10) and (5.11) are sufficient.

Using evaluation maps at infinity analogous to that used in the proof of Lemma 5.1 it is easy to check that $\mathcal{D}_\infty/\mathcal{D} \cong U(1)$ and $\mathcal{P}_\infty/\mathcal{P} \cong \tilde{U}(1) \times \mathbb{R}$. Thus we find that the asymptotic symmetry group is given by

$$\mathcal{B} = \frac{(\mathcal{D}_\infty \ltimes \mathcal{P}_\infty)/(\mathcal{D} \ltimes \mathcal{P})}{\text{ker } \rho_\infty/\text{ker } \rho} = \frac{U(1) \times \tilde{U}(1) \times \mathbb{R}}{\tilde{U}(1)} = U(1) \times \mathbb{R}. \tag{5.12}$$

Finally, we want to determine the symmetry group $\mathcal{S}$, which requires evaluation of the quotient

$$\mathcal{S}^D = \mathcal{D}_\infty/\mathcal{D}_0. \tag{5.13}$$

We know that $\mathcal{D}/\mathcal{D}_0 = \text{Map}(S^\infty_{g,n})$, and first give a heuristic description[5] of $\mathcal{S}^D$, which will then be justified by a rigorous proof. Consider the effect of a diffeomorphism which is a rotation at the boundary on the generators of the fundamental group. If the angle of the rotation exceeds 2π, the diffeomorphism wraps the curves representing the generators around the boundary at infinity as depicted in Fig. 4.

[5]We thank D. Giulini, for pointing out this part of the argument to CM

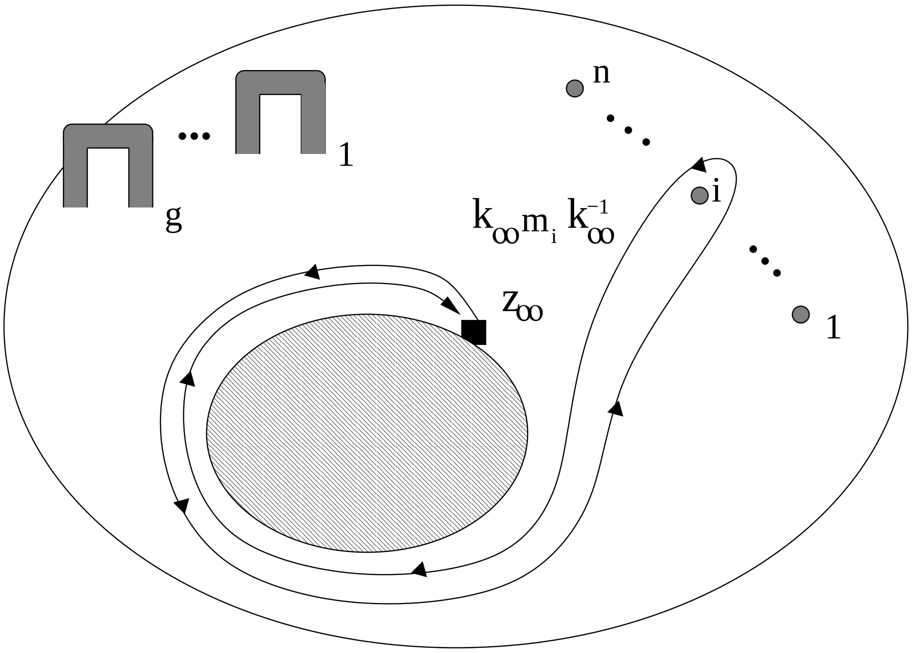

Fig. 4
The action of a rotation at the boundary

It then acts nontrivially on the fundamental group by conjugating every element with the element

$$k_\infty = b_g a_g^{-1} b_g^{-1} a_g \cdots b_1 a_1^{-1} b_1^{-1} a_1 m_n \cdots m_1. \tag{5.14}$$

We can thus identify it with a combination of a large, asymptotically trivial diffeomorphism C_∞ acting on the fundamental group by global conjugation with k_∞ and a small diffeomorphism that reduces to a asymptotic rotation by $\alpha - 2\pi$. Note that the element $[C_\infty] \in \mathrm{Map}(S^\infty_{g,n})$ is central, which follows from the fact that the equivalence class of the path k_∞ around the boundary is invariant under diffeomorphisms that keep the boundary fixed. Denoting by Ω the rotation by 2π in the cover $\tilde{U}(1)$ and writing $\mathbb{Z}$ for the central subgroup of $\mathrm{Map}(S^\infty_{g,n}) \times \tilde{U}(1)$ generated by $([C_\infty], \Omega)$, we claim

Lemma 5.2 *The quotient $\mathcal{S}^D = \mathcal{D}_\infty/\mathcal{D}_0$ is given by*

$$\mathcal{S}^D = (\mathrm{Map}(S^\infty_{g,n}) \times \tilde{U}(1))/\mathbb{Z}. \tag{5.15}$$

Proof Our proof closely follows the argument given in [13], [26]. We consider the nontrivial part of the exact sequence associated to the bundle structure

$$\begin{array}{ccc} \mathrm{Map}(S^\infty_{g,n}) \cong \mathcal{D}/\mathcal{D}_0 & \stackrel{i}{\to} & \mathcal{D}_\infty/\mathcal{D}_0 \cong \mathcal{S}^D \\ & & \downarrow p \\ & & \mathcal{D}_\infty/\mathcal{D} \cong U(1), \end{array} \tag{5.16}$$

which takes the form

$$1 \to \pi_1(\mathcal{D}_\infty/\mathcal{D}_0) \xrightarrow{p_*} \pi_1(U(1)) \cong \mathbb{Z} \xrightarrow{\partial_*} \mathrm{Map}(S^\infty_{g,n}) \xrightarrow{i_*} \mathcal{S}^D/\mathcal{S}^D_0 \to 1, \tag{5.17}$$

where the function ∂_* is obtained as follows. We parametrise $\mathcal{D}_\infty/\mathcal{D} \cong U(1)$ by $e^{i\alpha}$, $\alpha \in [0, 2\pi)$ and choose the loop $\Gamma(\alpha) = e^{i\alpha}$, $\alpha \in [0, 2\pi]$, whose homotopy class generates $\pi_1(U(1)) \cong \mathbb{Z}$. By assigning to every element $e^{i\alpha} \in U(1)$ the equivalence class of diffeomorphisms which are a global rotation of angle α at spatial infinity, we can define a curve $\bar{\Gamma}$ in $\mathcal{S}^D = \mathcal{D}_\infty/\mathcal{D}_0$ that starts at the identity. The projection p_* in (5.17) maps an equivalence class of diffeomorphisms in $\mathcal{S}^D$ to the value they take at spatial infinity. So we see that $p_* \circ \bar{\Gamma} = \Gamma$, and $\bar{\Gamma}$ is the lift of Γ to $\mathcal{S}^D$. Visualising the action of the mapping classes $\bar{\Gamma}(\alpha)$ by their action on a set of curves representing the generators of the fundamental group, we see that the curves are wrapped once around the boundary at spatial infinity as α approaches 2π (Fig. 4). Therefore, we have $\partial_*([\Gamma]) = \bar{\Gamma}(2\pi) = [C_\infty]$. It follows from the exactness of (5.17) that $\pi_1(\mathcal{S}^D) = \{1\}$ and $\mathcal{S}^D/\mathcal{S}^D_0 \cong \mathbb{Z}$. This means that the holonomy group of the bundle (5.16) is the central subgroup of $\mathrm{Map}(S^\infty_{g,n})$ generated by $[C_\infty]$ and that $\mathcal{S}^D$ is given by (5.15).□

Using Lemmas 5.1 and 5.2, we can evaluate the quotients occuring in (5.9) and obtain symmetry group

$$\mathcal{S} = \frac{\Big((\mathrm{Map}(S^\infty_{g,n}) \times \tilde{U}(1))/\mathbb{Z}\Big) \times \tilde{U}(1) \times \mathbb{R}}{\tilde{U}(1)} = \frac{\mathrm{Map}(S^\infty_{g,n}) \times \tilde{U}(1) \times \mathbb{R}}{\mathbb{Z}}. \tag{5.18}$$

The above discussion can be generalised from the centre of mass frame to a general frame. As explained at the beginning of Sect. 4, this is easily accommodated in the Chern-Simons formulation. We then allow general Poincaré transformations as asymptotic symmetries. Writing again $\mathbb{Z}$ for the central subgroup of $\mathrm{Map}(S^\infty_{g,n}) \times \tilde{P}^\uparrow_3$ generated by $([C_\infty], \exp(-2\pi J_0))$, the symmetry group is the quotient

$$\frac{\mathrm{Map}(S^\infty_{g,n}) \times \tilde{P}^\uparrow_3}{\mathbb{Z}}. \tag{5.19}$$

5.2 Hamiltonian, total angular momentum and conserved quantities

After determining the symmetry group of our model, we now investigate how the continuous symmetries at spatial infinity are related to conserved quantities. We show that the total momentum and total angular momentum three-vector of the universe generate asymptotic Poincaré transformations. In the centre of mass frame of the universe, only spatial rotations and time translations are admitted. The conserved quantities associated to these transformations are the total angular momentum and the total energy of the universe.

The total momentum and angular momentum three-vectors of the universe as seen by the observer are given by the holonomy h_∞ of the curve k_∞ (5.14) around the boundary pictured in Fig. 3

$$\begin{aligned} h_\infty &= (u_{tot}, \boldsymbol{a}_{tot}) = (\exp(-p^a_{tot}J_a), -\mathrm{Ad}(\exp(-p^a_{tot}J_a))\boldsymbol{j}_{tot}) \\ &= (B_g {A_g}^{-1} {B_g}^{-1} A_g) \cdots (B_1 {A_1}^{-1} {B_1}^{-1} A_1) M_n \cdots M_1, \end{aligned} \tag{5.20}$$

which gives

$$\begin{aligned}
u_{tot} &= \exp(-p^a_{tot}J_a) = u_{K_g}\cdots u_{K_1}u_{M_n}\cdots u_{M_1} && (5.21)\\
\boldsymbol{j}_{tot} &= \sum_{i=1}^{n}\mathrm{Ad}(u_{M_1}^{-1}\cdots u_{M_{i-1}}^{-1})\boldsymbol{j}^{M_i} && (5.22)\\
&+ \sum_{i=1}^{g}\Bigg(\mathrm{Ad}(u_{M_1}^{-1}\cdots u_{M_n}^{-1}u_{K_1}^{-1}\cdots u_{K_i}^{-1})\big((\mathrm{Ad}(u_{B_i})-\mathrm{Ad}(u_{B_i}u_{A_i}^{-1}))\boldsymbol{j}^{B_i}\\
&\qquad -\mathrm{Ad}(u_{M_1}^{-1}\cdots u_{M_n}^{-1}u_{K_1}^{-1}\cdots u_{K_i}^{-1})\big((\mathrm{Ad}(u_{B_i})-\mathrm{Ad}(u_{K_i}))\boldsymbol{j}^{A_i}\Bigg)
\end{aligned}$$

with $u_{K_j} := u_{B_j}u_{A_j}^{-1}u_{B_j}^{-1}u_{A_j}$ for $j=1,\ldots,g$. In general, the element $u_{tot}\in\tilde{L}_3^{\uparrow}$ does not have to be elliptic, but can also be parabolic or hyperbolic, which gives rise to interesting effects such as Gott-pairs [28]. In this article, we will restrict our interpretation to the elliptic case and assume $p^0_{tot}>0$, $\boldsymbol{p}^2_{tot}=\mu^2$ with $\mu\in(0,2\pi)$. We see that the total momentum and angular momentum three-vectors $\boldsymbol{p}_{tot}$, $\boldsymbol{j}_{tot}$ are *not* obtained by simply adding the corresponding quantities for the individual handles and particles. However, using the decoupled coordinates (4.20) defined in Sect. 4, the angular momentum of the universe can be rewritten as

$$\boldsymbol{j}_{tot} = \sum_{i=1}^{n}\boldsymbol{j}^{M_i'}+\sum_{i=1}^{g}\boldsymbol{j}^{H_i'} \qquad (5.23)$$

where $\boldsymbol{j}^{H_i'}$ is given by equation (4.24) and can be interpreted as the angular momentum associated to the i^{th} handle.

In order to calculate the Poisson brackets involving these quantities, we must determine how right and left multiplication of the generators of the fundamental group affect the total holonomy (5.20). If we define

$$\begin{aligned}
&\Psi : \mathcal{C}_1\times\ldots\mathcal{C}_n\times\left(\tilde{P}_3^{\uparrow}\right)^{2g}\to\tilde{P}_3^{\uparrow}\times\mathcal{C}_1\times\ldots\mathcal{C}_n\times\left(\tilde{P}_3^{\uparrow}\right)^{2g} && (5.24)\\
&\Psi(M_1,\ldots,B_g)=(h_\infty,M_1,\ldots,B_g),
\end{aligned}$$

the derivative of the map Ψ induces a map of the vector fields associated to the generators. Expressing the image of the right and left invariant vector fields on $\mathcal{C}_1\times\ldots\mathcal{C}_n\times\left(\tilde{P}_3^{\uparrow}\right)^{2g}$ in terms of the right and left invariant vector fields on $\tilde{P}_3^{\uparrow}\times\mathcal{C}_1\times\ldots\mathcal{C}_n\times\left(\tilde{P}_3^{\uparrow}\right)^{2g}$ and inserting them in the Poisson bivector (4.14), we obtain the image of the Poisson bivector

$$\begin{aligned}
\Psi_*(B_{FR}) \quad &= B_{FR}+\tfrac{1}{2}r^{\alpha\beta}(L^{tot}_\alpha\wedge L^{tot}_\beta+R^{tot}_\alpha\wedge R^{tot}_\beta)+r^{\alpha\beta}R^{tot}_\alpha\wedge L^{tot}_\beta && (5.25)\\
&+r^{\alpha\beta}R^{tot}_\alpha\wedge\big(\big(\textstyle\sum_{i=1}^{n}R^{M_i}_\beta+L^{M_i}_\beta\big)+\big(\sum_{i=1}^{g}R^{A_i}_\beta+L^{A_i}_\beta+R^{B_i}_\beta+L^{B_i}_\beta\big)\big)\\
&+r^{\alpha\beta}\big(\big(\textstyle\sum_{i=1}^{n}R^{M_i}_\alpha+L^{M_i}_\alpha\big)+\big(\sum_{i=1}^{g}R^{A_i}_\alpha+L^{A_i}_\alpha+R^{B_i}_\alpha+L^{B_i}_\alpha\big)\big)\wedge L^{tot}_\beta,
\end{aligned}$$

where L^{tot} and R^{tot} are the right and left invariant vector fields associated to the element h_∞. With the r-matrix $r=P_a\otimes J^a$ and expressions (3.13) for the right and left invariant vector fields, we derive the Poisson brackets

$$\{j^a_{tot}, j^b_{tot}\} = -\epsilon^{abc} j^{tot}_c \qquad \{p^a_{tot}, j^b_{tot}\} = -\epsilon^{abc} p^{tot}_c \qquad \{p^a_{tot}, p^b_{tot}\} = 0 \tag{5.26}$$

$$\begin{aligned} \{j^a_{tot}, p^b_X\} &= -\epsilon^{abc} p^X_c \\ \{j^a_{tot}, j^b_X\} &= -\epsilon^{abc} j^X_c \end{aligned} \tag{5.27}$$

$$\begin{aligned} \{p^a_{tot}, p^b_X\} &= 0 \\ \{p^a_{tot}, j^b_X\} &= -\left((1-\mathrm{Ad}(u_X^{-1}))\cdot\frac{1}{1-\mathrm{Ad}(u_{tot}^{-1})}\right)^b_{\ c} \epsilon^{acd} p^{tot}_d = (1-\mathrm{Ad}(u_X^{-1}))^b_{\ c} (T^{-1}(\boldsymbol{p}_{tot}))^{ca}, \end{aligned} \tag{5.28}$$

where $X \in \{M_1 \ldots, B_g\}$ and $T(\boldsymbol{p}_{tot})$ is given by equation (3.18), its inverse by (3.14). We see that the Poisson brackets of the components of the total momentum and the total angular momentum three-vector are those of a free massive particle, as required. Together with the brackets (5.27) they confirm the interpretation of $\boldsymbol{j}_{tot}$ as the universe's total angular momentum three-vector.

Furthermore, the brackets (5.27) show that the components of $\boldsymbol{j}_{tot}$ are the infinitesimal generators of asymptotic Poincaré transformations. Boosts and rotations of the universe with respect to the reference frame of the observer act on the group elements associated to the particles and handles via global conjugation. For an infinitesimal boost or rotation we get

$$(u_X, \boldsymbol{a}_X) \to (\exp(\delta_a J^a), 0)(u_X, \boldsymbol{a}_X)(\exp(-\delta_a J^a), 0)$$

$$\Rightarrow \qquad \begin{aligned} \delta p^b_X &= \delta^a (J_a)^b_{\ c} p^c_X = -\delta_a \epsilon^{abc} p^X_c = \delta_a \{j^a_{tot}, p^b_X\} \\ \delta j^b_X &= \delta^a (J_a)^b_{\ c} j^c_X = -\delta_a \epsilon^{abc} j^X_c = \delta_a \{j^a_{tot}, j^b_X\}. \end{aligned}$$

The interpretation of the action (5.28) of the total momentum three-vector $\boldsymbol{p}_{tot}$ is less obvious. It should be expected that its components generate infinitesimal translations of the matter in the universe with respect to the observer. However, it turns out that these are not translations of the position coordinate $\boldsymbol{x}$ but related to the alternative position coordinate $\boldsymbol{q}$ (3.22) introduced in Sect. 3. A translation by $\boldsymbol{y}$ acts on the group elements of particles and handles via

$$(u_X, \boldsymbol{a}_X) \to (1, \boldsymbol{y})(u_X, \boldsymbol{a}_X)(1, -\boldsymbol{y})$$

$$\Rightarrow \qquad \begin{aligned} \delta p^b_X &= 0 \\ \delta j^b_X &= (1-\mathrm{Ad}(u_X^{-1}))^{bc} y_c, \end{aligned}$$

and by comparison with (5.28) we get

$$\begin{aligned} \delta p^b_X &= \delta_a \{p^a_{tot}, p^b_X\} \\ \delta j^b_X &= \delta_a \{p^a_{tot}, j^b_X\} \qquad\qquad \text{with } y^a = \left(T^{-1}(\boldsymbol{p}_{tot})\right)^{ab} \delta_b. \end{aligned}$$

Recalling the definition (3.22) of the alternative position coordinate $\boldsymbol{q}$, we see that this corresponds to the infinitesimal change of $\boldsymbol{x}$ induced by an infinitesimal change in $\boldsymbol{q}$

$$q^a \;\to\; q^a + \delta^a \tag{5.29}$$
$$x^a \;\to\; x^a + \left(T^{-1}(\boldsymbol{p})\right)^a{}_b\, \delta^b, \tag{5.30}$$

only that now the map $T(\boldsymbol{p}_{tot})$ occurring in the formula corresponds to the total momentum three-vector $\boldsymbol{p}_{tot}$ of the universe instead of the momentum three-vector of the particle. Note that for an infinitesimal translation in the direction of the total three-momentum $\boldsymbol{p}_{tot}$ we have $\delta^a = y^a$ and get the familiar transformation properties.

The time-component of the three-vector $\boldsymbol{p}^{tot}$ has the interpretation of the total energy of the universe as measured by the observer. It is the Hamiltonian of the system[6] and generates its time development

$$H = p^0_{tot}. \tag{5.31}$$

As it commutes with all three-momenta, the total three momentum of the universe as well as all individual three-momenta of particles and handles are conserved quantities. This reflects the fact that the interaction in (2+1)-dimensional gravity is topological, not dynamical in nature. Other conserved quantities are the zero-component of the total angular momentum three-vector and the total mass μ and spin s of the universe, given by

$$\boldsymbol{p}^2_{tot} = \mu^2 \qquad\qquad \boldsymbol{p}_{tot}\cdot\boldsymbol{j}_{tot} = \mu s. \tag{5.32}$$

The reference frame in which the centre of mass of the universe is at rest at the origin is characterised by the conditions

$$\boldsymbol{p}_{tot} = (H, 0, 0) \qquad \text{and} \qquad \boldsymbol{j}_{tot} = (j^0_{tot}, 0, 0), \tag{5.33}$$

where $H = \mu$ is the total energy or mass of the universe and $j^0_{tot} = s$ its total angular momentum or spin. The Hamiltonian is the conserved quantity associated to translations in the time direction. The spin of the universe, which can easily be seen to Poisson commute with the Hamiltonian, is the conserved quantity associated to asymptotic spatial rotations. The Hamilton equations for the total momentum and angular momentum three-vector in the momentum rest frame of the universe are obtained from equation (5.26)

$$\dot{p}^a_{tot} = \{H, p^a_{tot}\} = 0 \qquad \dot{j}^a_{tot} = \{H, j^a_{tot}\} = 0. \tag{5.34}$$

As expected, in the reference frame where the centre of mass of the universe is at rest at the origin, the universe's total momentum and total angular momentum three-vector have no time development and stay of the form (5.33).

[6] In a system with constraints, the Hamiltonian is only determined up to a linear combination of the constraints. However, as the constraints (4.12) Poisson commute with all functions on the phase space, their contribution is trivial and can be omitted.

5.3 The action of the mapping class group

We know from Sect. 4.2 that the elements of the mapping class group $\mathrm{Map}(S^\infty_{g,n})$ act on the phase space (4.10) via automorphisms of the fundamental group. However, we have yet to determine how this action affects the Poisson structure.

For explicit calculations we need to choose a set of generators of the mapping class group. This is best done in two steps. First we consider the pure mapping class group $\mathrm{PMap}(S^\infty_{g,n})$ which, by definition, is the subgroup of $\mathrm{Map}(S^\infty_{g,n})$ which leaves each puncture fixed. As explained in [24], it is generated by Dehn twists along a set of curves on the surface. For our purposes it is most convenient to work with the set of generators given by Schomerus and Alekseev [11]. This set is obtained from the homotopy classes of the curves a_i, δ_i, α_i, ϵ_i and $\kappa_{\nu\mu}$ pictured in Fig. 5.

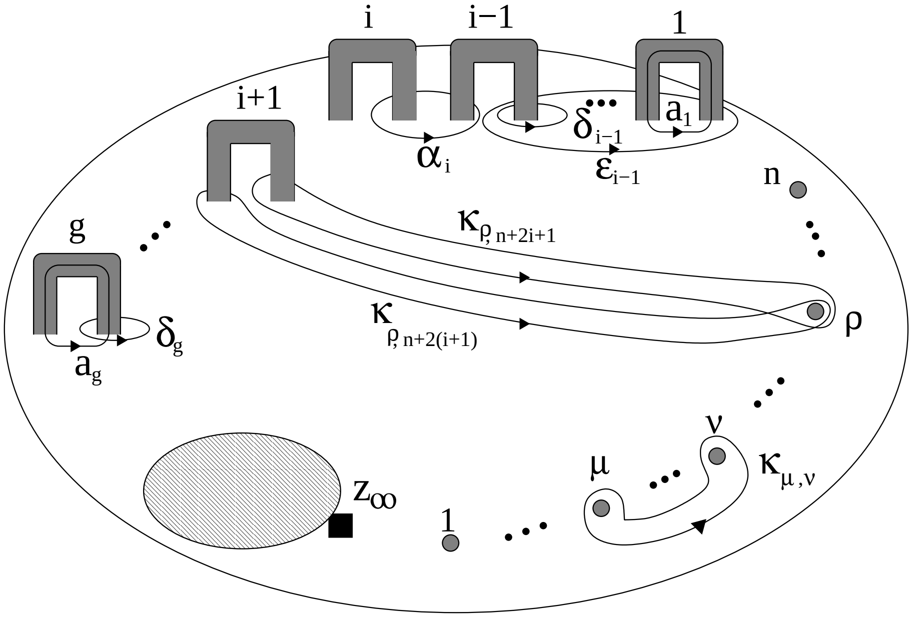

Fig. 5

The curves associated to the generators of the (pure) mapping class group

In terms of the generators of the fundamental group they are given by

$$\begin{array}{lll} a_i & i=1,\ldots,g & (5.35)\\ \delta_i = a_i^{-1}b_i^{-1}a_i & i=1,\ldots,g & \\ \alpha_i = a_i^{-1}b_i^{-1}a_ib_{i-1} & i=2,\ldots,g & \\ \epsilon_i = a_i^{-1}b_i^{-1}a_i\cdot(b_{i-1}a_{i-1}^{-1}b_{i-1}^{-1}a_{i-1})\cdot\ldots\cdot(b_1a_1^{-1}b_1^{-1}a_1) & i=2,\ldots,n & \\ \kappa_{\nu,\mu} = m_\mu m_\nu & 1\leq\nu<\mu\leq n & \\ \kappa_{\nu,n+2i-1} = a_i^{-1}b_i^{-1}a_im_\nu & \nu=1,\ldots,n,\ i=1,\ldots,g & \\ \kappa_{\nu,n+2i} = b_im_\nu & \nu=1,\ldots,n,\ i=1,\ldots,g\ . & \end{array}$$

Define a Dehn twist around each of these curves by embedding a small annulus around the curve and twisting its ends by an angle of 2π as shown in Fig. 6.

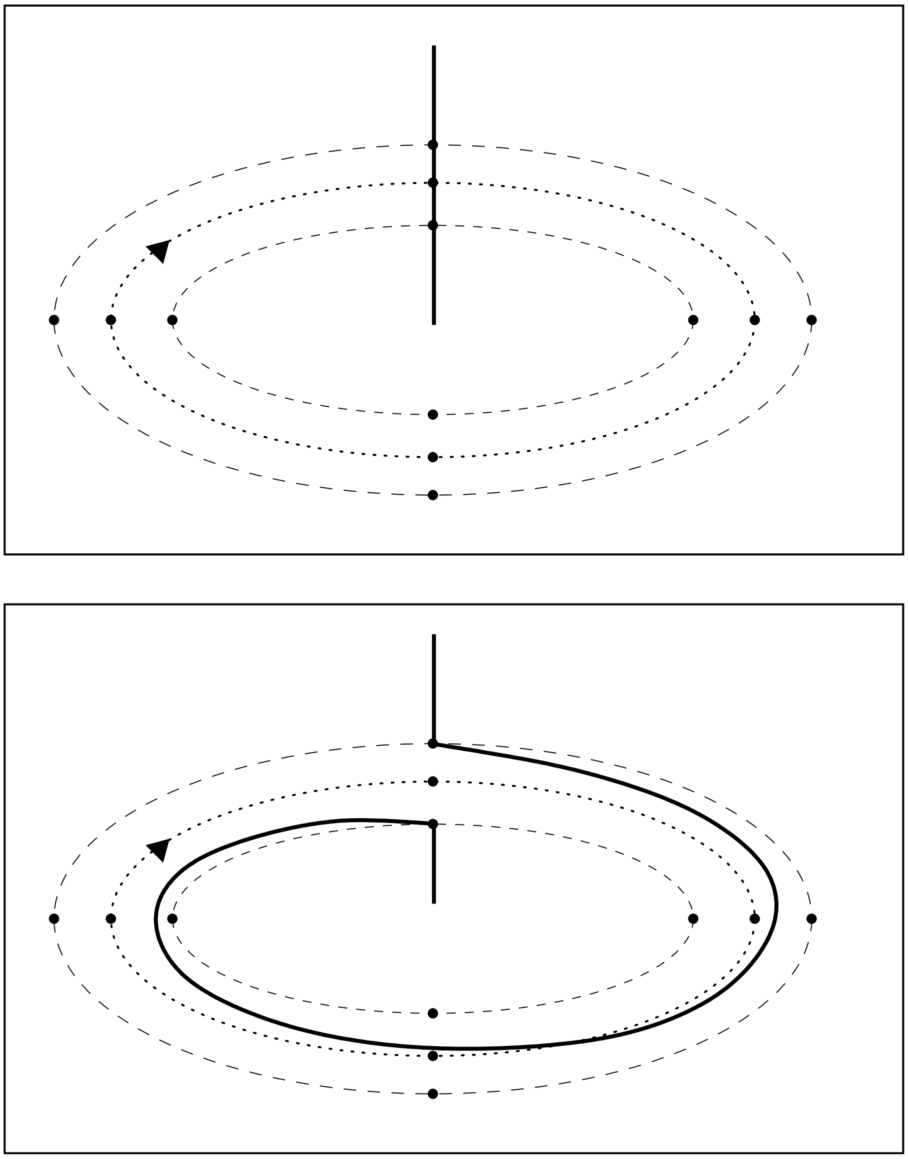

Fig. 6

The effect of a Dehn twist around an oriented loop (dotted line) on a curve intersecting the loop transversally (full line)

A Dehn twist around a given curve then defines an outer automorphism of the fundamental group, affecting only those elements whose representing curves intersect with it. Drawing the images of these curves as indicated in Fig. 6, we obtain the action of the pure mapping class group on the fundamental group of the surface. The associated action on the holonomies is given by the following table, where for each transformation we list only the holonomies that do not transform trivially

$$a_i : B_i \to B_i A_i \tag{5.36}$$

$$\delta_i : A_i \to B_i A_i \tag{5.37}$$

$$\begin{aligned} \alpha_i :& A_i \to B_i^{-1} A_i B_{i-1} = A_i \alpha_i \\ & B_{i-1} \to B_{i-1}^{-1} A_i^{-1} B_i A_i B_{i-1} A_i^{-1} B_i^{-1} A_i B_{i-1} = \alpha_i^{-1} B_{i-1} \alpha_i \\ & A_{i-1} \to B_{i-1}^{-1} A_i^{-1} B_i A_i A_{i-1} = \alpha_i^{-1} A_{i-1} \end{aligned} \tag{5.38}$$

$$\begin{aligned} \epsilon_i :& A_i \to B_i^{-1} A_i K_{i-1} \ldots K_1 = A_i \epsilon_i \qquad\qquad K_j := B_j A_j^{-1} B_j^{-1} A_j \\ & A_k \to K_1^{-1} \ldots K_{i-1}^{-1} (A_i^{-1} B_i A_i) A_k (A_i^{-1} B_i^{-1} A_i) K_{i-1} \ldots K_1 = \epsilon_i^{-1} A_k \epsilon_i \quad \forall 1 \le k < i \\ & B_k \to K_1^{-1} \ldots K_{i-1}^{-1} (A_i^{-1} B_i A_i) B_k (A_i^{-1} B_i^{-1} A_i) K_{i-1} \ldots K_1 = \epsilon_i^{-1} B_k \epsilon_i \quad \forall 1 \le k < i \end{aligned} \tag{5.39}$$

$$\begin{aligned} \kappa_{\nu\mu} :& M_\nu \to M_\nu^{-1} M_\mu^{-1} M_\nu M_\mu M_\nu = \kappa_{\nu\mu}^{-1} M_\mu \kappa_{\nu\mu} \\ & M_\mu \to M_\nu^{-1} M_\mu M_\nu = \kappa_{\nu\mu}^{-1} M_\mu \kappa_{\nu\mu} \\ & M_\kappa \to M_\nu^{-1} M_\mu^{-1} M_\nu M_\mu M_\kappa M_\mu^{-1} M_\nu^{-1} M_\mu M_\nu \qquad \forall \nu < \kappa < \mu \end{aligned} \tag{5.40}$$

$$\begin{aligned} \kappa_{\nu,n+2i-1} :& M_\nu \to M_\nu^{-1} A_i^{-1} B_i A_i M_\nu A_i^{-1} B_i^{-1} A_i M_\nu = \kappa_{\nu,n+2i-1}^{-1} M_\nu \kappa_{\nu,n+2i-1} \\ & A_i \to B_i^{-1} A_i M_\nu = A_i \kappa_{\nu,n+2i-1} \\ & X_j \to M_\nu^{-1} A_i^{-1} B_i A_i M_\nu A_i^{-1} B_i^{-1} A_i X_j A_i^{-1} B_i A_i M_\nu^{-1} A_i^{-1} B_i^{-1} A_i M_\nu \\ & \quad = \kappa_{\nu,n+2i-1}^{-1} M_\nu \kappa_{\nu,n+2i-1} M_\nu^{-1} X_j M_\nu \kappa_{\nu,n+2i-1}^{-1} M_\nu^{-1} \kappa_{\nu,n+2i-1}, \\ & \qquad X_j \in \{M_{\nu+1}, \ldots, M_n, A_1, \ldots, B_{i-1}\} \end{aligned} \tag{5.41}$$

$$\begin{aligned} \kappa_{\nu,n+2i} :& M_\nu \to M_\nu^{-1} B_i^{-1} M_\nu B_i M_\nu = \kappa_{\nu,n+2i}^{-1} M_\nu \kappa_{\nu,n+2i} \\ & B_i \to M_\nu^{-1} B_i M_\nu = \kappa_{\nu,n+2i}^{-1} B_i \kappa_{\nu,n+2i} \\ & A_i \to M_\nu^{-1} B_i^{-1} A_i B_i^{-1} M_\nu^{-1} B_i M_\nu \\ & \quad = \kappa_{\nu,n+2i}^{-1} A_i B_i^{-1} M_\nu^{-1} \kappa_{\nu,n+2i} \\ & X_j \to M_\nu^{-1} B_i^{-1} M_\nu B_i X_j B_i^{-1} M_\nu^{-1} B_i M_\nu \\ & \quad = \kappa_{\nu,n+2i}^{-1} M_\nu B_i X_j B_i^{-1} M_\nu^{-1} \kappa_{\nu,n+2i}, \qquad X_j \in \{M_{\nu+1}, \ldots, M_n, A_1, \ldots B_{i-1}\}. \end{aligned} \tag{5.42}$$

The full mapping class group of the surface $S_{g,n}^\infty$ is related to pure mapping class group by the short exact sequence

$$1 \to \mathrm{PMap}(S_{g,n}^\infty) \xrightarrow{i} \mathrm{Map}(S_{g,n}^\infty) \xrightarrow{\pi} S_n \to 1, \tag{5.43}$$

where i is the canonical embedding and π is the projection onto the permutation group defined in (4.9). It follows that we obtain a set of generators for the full mapping class group by supplementing the set of generators of the pure mapping class group with a set of generators σ_i which get mapped to the elementary transpositions via π. As explained in [24],[25] the σ_i generate the braid group on the surface $S^\infty_{g,n}$. The action of these generators on the holonomies around the punctures is shown in Fig. 7.

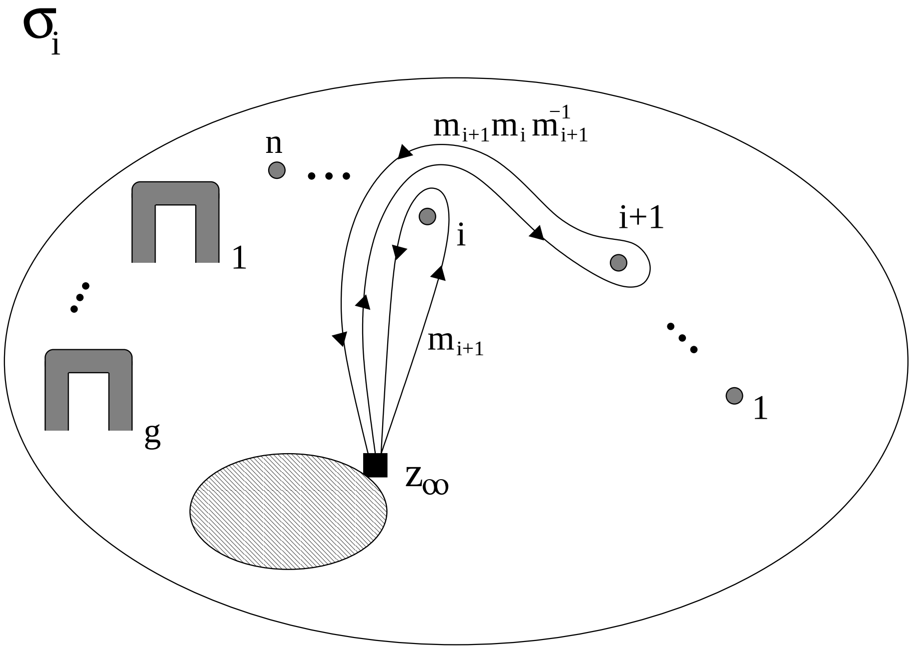

Fig. 7
Generators of the braid group on the surface $S^\infty_{g,n}$

They act on the phase space variables M_i according to

$$\begin{aligned} \sigma_i : & M_i \to M_{i+1} \\ & M_{i+1} \to M_{i+1} M_i M_{i+1}^{-1} \, . \end{aligned} \tag{5.44}$$

Armed with the expressions (5.36) to (5.44) for the action of the (full) mapping class group on the generators of the holonomies we can prove

Theorem 5.3 *The mapping class group Map($S^\infty_{g,n}$) acts on the phase space via Poisson isomorphisms.*[7]

Proof The action of the Dehn twists (5.36) to (5.44) on the holonomies induces a transformation of the associated left and right invariant vector fields. This transformation can be calculated explicitly, allowing one to express the images of the vector fields in terms of the left and right invariant vector fields associated to the images of the holonomies. We demonstrate the general pattern on the case of the action (5.44) of the generators σ_i.

[7] A similar claim for a quantised version of the holonomies was made by Schomerus and Alekseev in [11]. A proof was announced in [11], but never published.

Determining how left and right multiplication of the holonomies affect their images under (5.44), we obtain an expression for the image of right and left invariant vector fields in terms of the vector fields associated to the images of the holonomies:

$$\begin{aligned} R^{M_i} &\to \mathrm{Ad}(M_{i+1})R^{\sigma(M_{i+1})} = \mathrm{Ad}(\sigma(M_i))R^{\sigma(M_{i+1})} \\ L^{M_i} &\to \mathrm{Ad}(M_{i+1})L^{\sigma(M_{i+1})} = \mathrm{Ad}(\sigma(M_i))L^{\sigma(M_{i+1})} \\ R^{M_{i+1}} &\to R^{\sigma(M_i)} - \mathrm{Ad}(M_{i+1})(R^{\sigma(M_{i+1})} + L^{\sigma(M_{i+1})}) \\ &\qquad = R^{\sigma(M_i)} - \mathrm{Ad}(\sigma(M_i))(R^{\sigma(M_{i+1})} + L^{\sigma(M_{i+1})}) \\ L^{M_{i+1}} &\to L^{\sigma(M_i)} + R^{\sigma(M_{i+1})} + L^{\sigma(M_{i+1})} \;. \end{aligned} \tag{5.45}$$

Note that the sum of all the vector fields that are affected by this transformation is form-invariant

$$R^{M_i} + L^{M_i} + R^{M_{i+1}} + L^{M_{i+1}} \to R^{\sigma(M_i)} + L^{\sigma(M_i)} + R^{\sigma(M_{i+1})} + L^{\sigma(M_{i+1})} \;, \tag{5.46}$$

implying form-invariance of the sum $R^{M_i} + L^{M_i} + \cdots + R^{B_g} + L^{B_g}$. This is not particular to transformation (5.44) but a general pattern for all the transformations (5.36) to (5.44).

To prove that the Dehn twists act as Poisson isomorphisms, we have to show that the Poisson bivector (4.14) is form-invariant. This is most easily seen by writing the Poisson bivector in the form (2.23), as a sum of a term tangential to the gauge orbits, which depends only on the antisymmetric part of the r-matrix, and a term transversal to them, involving only on the matrix $t^{\alpha\beta}$ representing the bilinear form. Now recall Fock and Rosly's description of the moduli space [6] and interpret the fundamental group as a graph. Both, the original and the transformed graph could be used to describe the Poisson structure on the moduli space and should therefore be identical up to gauge transformations, which in this case are realised as global conjugations at the vertex. However, as the part of the Poisson bivector tangential to the gauge orbits is proportional to the sum $R^{M_i} + L^{M_i} + \cdots + R^{B_g} + L^{B_g}$, its form-invariance can be seen immediately from expression (2.23).

An explicit proof of the form-invariance of the part transversal to the gauge orbits proceeds as follows. We replace the vector fields occurring in the Poisson bivector by their images (5.45) and simplify the resulting expression using the relations between right and left invariant vector fields and the Ad-invariance of the Matrix $t^{\alpha\beta}$ representing the bilinear form. In the case of the transformation (5.44), this yields

$$\begin{aligned} &t^{\alpha\beta}\Big(R^{M_i}_\alpha \wedge L^{M_i}_\beta + R^{M_{i+1}}_\alpha \wedge L^{M_{i+1}}_\beta + (R^{M_i}_\alpha + L^{M_i}_\alpha) \wedge (R^{M_{i+1}}_\beta + L^{M_{i+1}}_\beta)\Big) \to \\ &t^{\alpha\beta}\Big(\big(\mathrm{Ad}(\sigma(M_i))R^{\sigma(M_{i+1})}\big)_\alpha \wedge \big(\mathrm{Ad}(\sigma(M_i))L^{\sigma(M_{i+1})}\big)_\beta \\ &\qquad\qquad + R^{\sigma(M_i)}_\alpha \wedge \big(L^{\sigma(M_i)} + R^{\sigma(M_{i+1})} + L^{\sigma(M_{i+1})}\big)_\beta \\ &\qquad\qquad + \big(\mathrm{Ad}(\sigma(M_i))(R^{\sigma(M_{i+1})} + L^{\sigma(M_{i+1})})\big)_\alpha \wedge R^{\sigma(M_i)}_\beta \Big) \\ &= t^{\alpha\beta}\left(R^{\sigma(M_i)}_\alpha \wedge L^{\sigma(M_i)}_\beta + R^{M'_{i+1}}_\alpha \wedge L^{M'_{i+1}}_\beta + (R^{\sigma(M_i)}_\alpha + L^{\sigma(M_i)}_\alpha) \wedge (R^{\sigma(M_{i+1})}_\beta + L^{\sigma(M_{i+1})}_\beta)\right), \end{aligned}$$

demonstrating the form-invariance of the Fock-Rosly bivector. □

6 Conclusion and outlook

In this paper we have given an explicit description of the phase space of the Chern-Simons formulation of (2+1)-dimensional gravity on a surface with with massive, spinning particles and a connected boundary component representing spatial infinity. This description provides a framework in which the physical properties of the phase space - its Poisson structure, the action of symmetries, conserved quantities - can be adressed in a rigorous and systematic way.

Mathematically, our approach is based on the combinatorial description of the moduli space of flat connections developed by Fock and Rosly. However, our main motivation was to give a description of the phase space of (2+1) gravity which allows one to investigate its *physical* content. The various choices we had to make in constructing our model - the r-matrix, the graph consisting of generators of the fundamental group, the treatment of the boundary conditions - were all determined by that motivation. It turned out that the physical requirements could be accommodated very easily in the Fock-Rosly framework and, conversely, that the mathematical quantities and concepts of this framework have a clear physical interpretation. Further evidence that our approach is well adapted to the physics of (2+1) gravity comes from the fact that it can readily be extended to include a cosmological constant.

Although we have endeavoured to relate our phase space to the metric formulation of (2+1) gravity as closely as possible, the precise relationship between the phase space (4.10) and, for example, the phase space of n particles (but no handles) discussed from the point of view of the metric formulation in [3] remains an open and important question. As we have indicated in our discussion of the single particle in Sect. 3, it is not difficult to relate the coordinates used in the two approaches and to show that, for a spinless particle, our symplectic structure reduces to that found in [12]. It would be interesting to check whether our symplectic structure similarly generalises that found for n spinless particles in [3].

Understanding the global relationship between the two phase spaces poses a more difficult challenge. A key issue here is the invertability of the dreibein, which is required in Einstein's metric formulation of gravity but not in the Chern-Simons formulation. Matschull argues in [23] that this leads to an identification of states in the Chern-Simons formulation that would not be identified in the metric formulation, implying different phase spaces for the two formulations. He illustrates his claim with an example of two physically distinct spacetimes, which, according to him, should be identified in the Chern-Simons formulation. However, an important ingredient of our phase space (4.10) is the parametrisation in terms of holonomies together with a label for the set of curves along which the holonomies are to be calculated. By insisting that the description of the same connection in terms of different sets of generators of the fundamental group leads to physically distinct states we are able to distinguish between the spacetimes described in [23]. In our formalism, they are related by an element of the mapping class group (for particles on a disc without handles this is just the braid group) and therefore not identified. By taking sufficient care in the interpretation of our phase space parameters it thus seems possible to extract spacetime physics from our formalism which is in agreement with the Einstein's metric formulation of gravity. One purpose of the present paper is to provide a firm foundation for such an investigation.

Another important issue to be adressed in further investigations is the question of gauge invariant observables. Nelson and Regge studied this question for closed surfaces of various genera, giving a presentation of the classical algebra of observables and discussing its quantisation [1]. Martin investigated the case of a sphere with an arbitrary number of punctures representing massive particles with spin [31]. By adapting our description of the phase space to the case of closed surfaces, we should be able to determine how it is related to these articles. It would be interesting to see if this allows one to treat the case of a surface without punctures and the punctured sphere in a common framework.

To end, we stress that the description of phase space based on the work of Fock and Rosly is a particularly convenient starting point for quantisation. Fock and Rosly's description of the moduli space of flat G connections was developped further by Alekseev, Grosse and Schomerus, who invented a quantisation procedure, the combinatorial quantisation of Chern-Simons theories, based on this description [9, 10, 11]. In this procedure, which has been worked out fully for the case of compact semisimple Lie groups and applied to the non-compact group $SL(2,\mathbb{C})$ in [32], quantum groups play a key role. In [14], the relevant quantum group for the Poincaré group and the r-matrix (3.12) used in this paper was identified as the quantum double of the Lorentz group. Moreover, it was explained how to reconstruct the two-particle Hilbert space and the scattering cross section of two massive particles from the representation theory and the R-matrix of the Lorentz double. This suggests that an extension of this approach should enable one to quantise the phase n-massive particles on a genus g surface discussed in the present paper.

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A Classical r-matrices and Poisson Lie groups

This appendix summarises briefly some facts about Poisson-Lie groups used in this article. For a more complete and mathematically exact treatment we refer the reader to the book by Chari and Pressley [29] and the articles by Alekseev and Malkin [7],[8] and Semenov-Tian-Shansky [30].

A.1 Lie bialgebras and Poisson-Lie groups

Poisson-Lie groups and Lie bialgebras

A *Poisson-Lie group* is a Lie group G that is also a Poisson manifold such that the multiplication map $\mu : G \times G \to G$ is a Poisson map with respect to the Poisson structure on G and the direct product Poisson structure on $G \times G$. Just as a Lie algebra is the infinitesimal object associated to a Lie group, there is an infinitesimal object, the *Lie bialgebra*, associated to a Poisson-Lie group. A Lie bialgebra is a Lie algebra $\mathfrak{g}$ with a skew symmetric map $\delta : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$, the *cocommutator*, such that $\delta^* : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$ defines a Lie bracket on its dual space $\mathfrak{g}^*$ and δ is a one-cocycle of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$:

$$\delta([X,Y]) = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)\delta(Y) - (\mathrm{ad}_Y \otimes 1 + 1 \otimes \mathrm{ad}_Y)\delta(X) \quad \forall X, Y \in \mathfrak{g}. \tag{A.1}$$

Every Poisson-Lie group G has a unique *tangent Lie bialgebra* $\mathfrak{g}$. Its cocommutator is defined as the dual of the commutator on $\mathfrak{g}^*$ given by $[\xi_1, \xi_2]_{\mathfrak{g}^*} := d_e\{f_1, f_2\}$, where f_1, f_2 are any two smooth functions on G with $d_e f_1 = \xi_1$, $d_e f_2 = \xi_2 \in \mathfrak{g}^*$. (The bracket $[\,,\,]_{\mathfrak{g}^*}$ does not depend on the choice of these functions.) Conversely, for every Lie bialgebra $\mathfrak{g}$ there is a unique connected and simply connected Poisson-Lie group with tangent Lie bialgebra $\mathfrak{g}$.

Duals and doubles

As can be inferred from the definitions above, for every Lie bialgebra $\mathfrak{g}$ its dual space $\mathfrak{g}^*$ is also a Lie bialgebra. Its commutator is the dual of the cocommutator on $\mathfrak{g}$ and its cocommutator is the dual of the commutator on $\mathfrak{g}$. The unique connected and simply connected Poisson-Lie group G^* with tangent Lie bialgebra $\mathfrak{g}^*$ is called the *dual* of G.

A Lie bialgebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ can be combined into a larger Lie bialgebra, its *classical double* $\mathcal{D}(\mathfrak{g})$. As a Lie algebra, it is the direct sum of the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$, and its cocommutator is given by

$$\delta_{\mathcal{D}(\mathfrak{g})}(u) = (\mathrm{ad}_u \otimes 1 + 1 \otimes \mathrm{ad}_u)(r) \qquad \forall u \in \mathfrak{g} \oplus \mathfrak{g}^*, \tag{A.2}$$

where $r \in \mathfrak{g} \otimes \mathfrak{g}^* \subset \mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(\mathfrak{g})$ is the element associated to the identity map $\mathrm{id} : \mathfrak{g} \to \mathfrak{g}$ via the canonical pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$. The unique connected and simply connected Poisson-Lie group $\mathcal{D}(G)$ with tangent Lie bialgebra $\mathcal{D}(\mathfrak{g})$ is called the *double* of the Poisson-Lie group G. As a Lie group, the double is the product of G and its dual $\mathcal{D}(G) = G \times G^*$. With respect to a basis X_α, $\alpha = 1, \ldots, dim\,\mathfrak{g}$, of $\mathfrak{g}$ and the dual basis ξ^α, $\alpha = 1, \ldots, dim\,\mathfrak{g}$, of $\mathfrak{g}^*$, its Poisson structure is given by the Poisson bivector

$$B_{\mathcal{D}(G)} = L_{X_\alpha} \otimes L_{\xi^\alpha} - R_{X_\alpha} \otimes R_{\xi^\alpha} \tag{A.3}$$

where L_{X_α}, R_{X_α}, L_{ξ^α} and R_{ξ^α} denote the right and left invariant vector fields on $\mathcal{D}(G)$ associated to the basis X_α and its dual ξ^α.

Classical r-matrices, factorising and the Heisenberg double structure

The element r in (A.2) has the properties of a classical r-matrix. A *classical r-matrix* for a Lie algebra $\mathfrak{g}$ is a tensor $r \in \mathfrak{g} \otimes \mathfrak{g}$, whose symmetric part $r + \sigma(r)$ is non-degenerate and Ad-invariant and that satisfies the the classical Yang-Baxter equation (CYBE)

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{A.4}$$

where $r_{12} = r \otimes r \otimes 1$, $r_{13} = r \otimes 1 \otimes r$, $r_{23} = 1 \otimes r \otimes r$ and σ denotes the flip $\sigma(a \otimes b) = b \otimes a$.

A classical r-matrix allows one to construct an cocommutator for the Lie algebra $\mathfrak{g}$ via (A.2) and a Poisson-Lie structure for the Lie group G given in analogy to (A.3). In addition to this, it defines canonical maps between a Poisson-Lie group and its dual. Define the Lie algebra homomorphisms $\rho, \rho_\sigma : \mathfrak{g}^* \to \mathfrak{g}$

$$\langle \rho(\xi), \eta \rangle := \langle r, \xi \otimes \eta \rangle \qquad \langle \rho_\sigma(\xi), \eta \rangle := -\langle \sigma(r), \xi \otimes \eta \rangle \qquad \forall \xi, \eta \in \mathfrak{g}^*, \tag{A.5}$$

where $\langle\, ,\, \rangle$ denotes the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. They lift to Lie group homomorphisms $S, S_\sigma : G^* \to G$, defining a map $Z : G^* \to G$ with $Z(g) := S(g) S_\sigma(g)^{-1}$, which is a local diffeomorphism around the identity. If it is a global diffeomorphism, the Poisson-Lie group G is called *factorisable* and every $g \in G$ has a unique decomposition

$$g = g_+ g_-^{-1} \qquad \text{with} \qquad g_+ = S(Z^{-1}(g)) \quad \text{and} \quad g_- = S_\sigma(Z^{-1}(g)). \tag{A.6}$$

Using these maps between a Poisson-Lie group and its dual, we can embed G and G^* into $G \times G$

$$\begin{aligned} G \to G \times G : &\qquad h \to (h, h) &\qquad \forall h \in G \\ G^* \to G \times G : &\qquad L \to (S(L), S_\sigma(L)) &\qquad \forall L \in G^*. \end{aligned} \tag{A.7}$$

In [30], Semenov-Tian-Shansky introduced a Poisson structure on $G \times G$ which is such that these embeddings are Poisson maps with respect to the Poisson-Lie structures on G and G^*. The Lie group $G \times G$ with this Poisson structure is called the *Heisenberg double* $\mathcal{D}_+(G)$ of G (note that it is not a Poisson-Lie group). Explicitly, the Poisson structure is given by the bivector

$$\begin{aligned} B'_{\mathcal{D}_+(G)}(g) =& \tfrac{1}{2} r^{\beta\alpha} (R^1_{X_\alpha} \wedge R^1_{X_\beta} + R^2_{X_\alpha} \wedge R^2_{X_\beta} + L^1_{X_\alpha} \wedge L^1_{X_\beta} + L^2_{X_\alpha} \wedge L^2_{X_\beta}) \\ &+ r^{\beta\alpha} (R^1_{X_\alpha} \wedge R^2_{X_\beta} + L^1_{X_\alpha} \wedge L^2_{X_\beta}), \end{aligned} \tag{A.8}$$

where the indices 1 and 2 refer to the first and second argument of $G \times G$ and L_{X_α}, R_{X_α} again denote the right and left invariant vector fields associated to the basis X_α. It was shown in [33] that the Poisson structure on the Heisenberg double of a globally factorisable group is symplectic.

A.2 The Poisson-Lie group $\tilde{P}_3^\uparrow$

The Lie bialgebra $iso(2,1)$ as classical double

The Lie bialgebra $iso(2,1)$ with generators P_a, J_a, $a = 0, 1, 2$, is is the classical double of $so(2,1)$: Take $so(2,1)$ with generators J_a, $a = 0, 1, 2$, and commutator

$$[J_a, J_b] = \epsilon_{abc} J^c. \tag{A.9}$$

Define a dual basis P_a, $a = 0, 1, 2$, of the space $so(2,1)^*$ with canonical pairing

$$\langle J_a, P^b \rangle = \eta_{ab} \tag{A.10}$$

and equip it with the trivial Lie algebra structure

$$[P^a, P^b] = 0. \tag{A.11}$$

Then, the element

$$r = J_a \otimes P^a \qquad \text{and its flip} \qquad \sigma(r) = P^a \otimes J_a \tag{A.12}$$

are r-matrices for $so(2,1)$. The commutator on the classical double $iso(2,1) = so(2,1) \oplus so(2,1)^*$ is given by

$$[J_a, J_b] = \epsilon_{ab}{}^c J_c \qquad [J_a, P^b] = \epsilon_a{}^b{}_c P^c \qquad [P^a, P^b] = 0, \tag{A.13}$$

the cocommutator by

$$\delta(J_a) = 0 \qquad \delta(P^a) = \epsilon^a{}_{bc} P^b \otimes P^c. \tag{A.14}$$

The connected and simply connected Poisson-Lie group associated to this Lie bialgebra structure is the universal cover $\tilde{P}_3^\uparrow$ of the Poincaré group $P_3^\uparrow$, with the multiplication law (2.2).

The dual of $\tilde{P}_3^\uparrow$

The dual of Lie bialgebra $iso(2,1)$ is the Lie algebra generated by P^a, J_a, $a = 0, 1, 2$, with commutator

$$[J_a, J_b] = \epsilon_{ab}{}^c J_c \qquad [J_a, P^b] = 0 \qquad [P^a, P^b] = 0, \tag{A.15}$$

and cocommutator

$$\delta(J_a) = \epsilon_{ab}{}^c (P^b \otimes J_c - J_c \otimes P^b) \qquad \delta(P^a) = \epsilon^a{}_{bc} P^b \otimes P^c. \tag{A.16}$$

The unique connected and simply connected Poisson-Lie group $(\tilde{P}_3^\uparrow)^*$ associated to this Lie bialgebra is the direct product $\tilde{L}_3^\uparrow \times \mathbb{R}^3$ with group multiplication

$$(u, \boldsymbol{a})(u', \boldsymbol{a}') = (uu', \boldsymbol{a} + \boldsymbol{a}'). \tag{A.17}$$

Heisenberg double of $\tilde{P}_3^\uparrow$

The maps (A.5) ρ,$\rho_\sigma : iso(2,1)^* \to iso(2,1)$ from its dual into $iso(2,1)$ are given by

$$\rho(P_a) = 0 \qquad\qquad \rho(J_a) = J_a \tag{A.18}$$

$$\rho_\sigma(P_a) = -P_a \qquad\qquad \rho_\sigma(J_a) = 0. \tag{A.19}$$

Their covering maps $S,\, S_\sigma : \left(\tilde{P}_3^\uparrow\right)^* \to \tilde{P}_3^\uparrow$ are obtained using the exponential map

$$S(u,\boldsymbol{a}) = (u,0) \qquad S_\sigma(u,\boldsymbol{a}) = (1, -\mathrm{Ad}(u^{-1})\boldsymbol{a}) = (1,\boldsymbol{j}). \tag{A.20}$$

They allow to factorise every element $g \in \tilde{P}_3^\uparrow$ according to (A.6) as

$$g = (u,\boldsymbol{a}) = (u,\boldsymbol{a})_+ (u,\boldsymbol{a})_-^{-1} \tag{A.21}$$

with

$$(u,\boldsymbol{a})_+ = (u,0) \qquad \text{and} \qquad (u,\boldsymbol{a})_- = (1,-\mathrm{Ad}(u^{-1})\boldsymbol{a}) = (1,\boldsymbol{j}). \tag{A.22}$$

Denoting the first argument in $\tilde{P}_3^\uparrow \times \tilde{P}_3^\uparrow$ by A, the second by B and using the r-matrix r in (A.12), the Poisson bivector (A.8) on the Heisenberg double of $\tilde{P}_3^\uparrow$ becomes

$$\begin{aligned} B'_{\mathcal{D}_+(\tilde{P}_3^\uparrow)}(g) &= \tfrac{1}{2}(P_a^{A_R} \wedge J^a_{A_R} + P_a^{B_R} \wedge J^a_{B_R} + P_a^{A_L} \wedge J^a_{A_L} + P_a^{B_L} \wedge J^a_{B_L}) \\ &+ (P_a^{A_R} \wedge J^a_{B_R} + P_a^{A_L} \wedge J^a_{B_L}). \end{aligned} \tag{A.23}$$

After inserting the expressions (3.13) for the right and left invariant vector fields associated to the two arguments

$$\begin{aligned} P_a^{X_R} f(g_A,g_B) &= -\frac{\partial f}{\partial j^a_X}(g_A,g_B) \\ P_a^{X_L} f(g_A,g_B) &= \mathrm{Ad}(u_X)_{ab}\frac{\partial f}{\partial j_b^X}(g_A,g_B) \\ J_a^{X_R} f(g_A,g_B) &= \big(\frac{1}{\mathrm{Ad}(u_X)-1}\big)_{ab}\epsilon^b_{\;cd}p^c_X\frac{\partial f}{\partial p^d_X}(g_A,g_B) - \epsilon_{abc}j^b_X\frac{\partial f}{\partial j_c^X}(g_A,g_B) \\ J_a^{X_L} f(g_A,g_B) &= \big(\frac{\mathrm{Ad}(u_X)}{1-\mathrm{Ad}(u_X)}\big)_{ab}\epsilon_{bcd}p^c_X\frac{\partial f}{\partial p^d_X}(g_A,g_B) \qquad \text{for} \quad X = A,B \end{aligned} \tag{A.24}$$

and parametrising g_A and g_B as in (2.4), we obtain the Poisson bracket on $\mathcal{D}_+(\tilde{P}_3^\uparrow)$:

$$\begin{aligned} \{j^a_X, p^b_X\} &= -\left(\frac{1}{1-\mathrm{Ad}(u_X)}\right)^a_{\;d}\epsilon^{bcd}p_c^X & X &= A,B \\ \{j^a_X, j^b_X\} &= -\epsilon^{abc}j_c^X & X &= A,B \\ \{j^a_A, p^b_B\} &= -\left(\frac{1}{1-\mathrm{Ad}(u_B)} + \mathrm{Ad}(u_A^{-1})\frac{\mathrm{Ad}(u_B)}{1-\mathrm{Ad}(u_B)}\right)^a_{\;d}\epsilon^{bcd}p_c^B \\ \{j^a_A, j^b_B\} &= -\epsilon^{abc}j_c^B \\ \{j^a_B, p^b_A\} &= 0 \\ \{p^a_B, p^b_A\} &= 0. \end{aligned} \tag{A.25}$$

The Poisson structure on the Heisenberg double of $\tilde{P}_3^\uparrow$ is symplectic. We give a short, direct proof here.

Theorem A.1 *The Poisson structure* (A.25) *on the Heisenberg double of the Poisson-Lie group* $\tilde{P}_3^\uparrow$ *is symplectic.*

Proof: We have to show that the determinant of the Matrix representing the Poisson structure (A.25) with respect to a basis does not vanish. With respect to the (ordered) basis $B = (\frac{\partial}{\partial p_A^a}, \frac{\partial}{\partial j_A^a}, \frac{\partial}{\partial p_B^a}, \frac{\partial}{\partial j_B^a})$, $a = 0, 1, 2$, the Poisson bracket (A.25) is represented by the matrix

$$M = \begin{pmatrix} 0 & -T_A^{-1} & 0 & 0 \\ (T_A^{-1})^T & -\epsilon^A & -S & -\epsilon^B \\ 0 & S^T & 0 & -T_B^{-1} \\ 0 & -\epsilon^B & (T_B^{-1})^T & -\epsilon^B \end{pmatrix}$$

with

$$\begin{aligned} S^{ab} &= \left(\frac{1}{1-\mathrm{Ad}(u_B)} + \mathrm{Ad}(u_A^{-1})\frac{\mathrm{Ad}(u_B)}{1-\mathrm{Ad}(u_B)}\right)^a_{\ d} \epsilon^{bcd} p_c^B \\ \left(\epsilon^X\right)^{ab} &= \epsilon^{abc} j_c^X \\ \left(T_X^{-1}\right)^{ab} &= \left(\frac{1}{1-\mathrm{Ad}(u_X)}\right)^a_{\ d} \epsilon^{bcd} p_c^X \qquad\qquad \text{for } \ X = A, B\ . \end{aligned}$$

The determinant of this matrix is given by

$$\det M = \left(\det T_A^{-1}\right)^2 \left(\det T_B^{-1}\right)^2 .$$

Comparing the matrix T_X^{-1} with the transformation (3.18) in Sect. 3, which relates the two sets of position and angular momentum coordinates, we find that it is given by

$$\left(T_X^{-1}\right)_{ab} = \left(T^{-1}(-\boldsymbol{p}^X)\right)_{ab} \qquad \text{for } X = A, B. \tag{A.26}$$

As the transformation (3.18) is invertible, its determinant does not vanish and neither does the determinant of M, proving that the Poisson structure of $\mathcal{D}_+(\tilde{P}_3^\uparrow)$ is symplectic. $\square$

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