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S-duality in SU(3) Yang-Mills theory with non-abelian unbroken gauge group

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Abstract

It is observed that the magnetic charges of classical monopole solutions in Yang-Mills-Higgs theory with non-abelian unbroken gauge group H are in one-to-one correspondence with coherent states of a dual or magnetic group $\tilde{H}$. In the spirit of the Goddard-Nuyts-Olive conjecture this observation is interpreted as evidence for a hidden magnetic symmetry of Yang-Mills theory. $SU(3)$ Yang-Mills-Higgs theory with unbroken gauge group $U(2)$ is studied in detail. The action of the magnetic group on semi-classical states is given explicitly. Investigations of dyonic excitations show that electric and magnetic symmetry are never manifest at the same time: Non-abelian magnetic charge obstructs the realisation of electric symmetry and vice-versa. On the basis of this fact the charge sectors in the theory are classified and their fusion rules are discussed. Non-abelian electric-magnetic duality is formulated as a map between charge sectors. Coherent states obey particularly simple fusion rules, and in the set of coherent states S-duality can be formulated as an $SL(2,\mathbf{Z})$ mapping between sectors which leaves the fusion rules invariant.

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1. Outline of the problem

Yang-Mills-Higgs theory with gauge group G spontaneously broken to a subgroup H contains two sorts of particles. The perturbative particles in the quantised theory can be organised into unitary irreducible representations (UIR's) of H and, in terminology borrowed from the abelian case $H = U(1)$, the particles in non-trivial UIR's may be called "electrically charged". It is well-known that spontaneously broken Yang-Mills-Higgs theory also contains magnetically charged particles if $\pi_2(G/H)$ is non-trivial. These arise as solitonic solutions of the classical Euler-Lagrange equations. As a consequence of the generalised Dirac quantisation condition the magnetic charges are quantised and take values in a certain lattice [1]. It was emphasised by Goddard, Nuyts and Olive (GNO) [2] that one could interpret this lattice as the weight lattice of a dual or magnetic group $\tilde{H}$. GNO conjectured that the presence of the magnetic monopoles signals a hidden magnetic symmetry of Yang-Mills theory, and that the full symmetry is the product $\tilde{H} \times H$. This conjecture was further elaborated by Montonen and Olive [3] in the case where $H = U(1)$. According to the Montonen-Olive electric-magnetic duality conjecture the physics of the electrically charged particles at coupling e is the same as that of the magnetically charged particles at coupling $4\pi/e$. More recently the generalisation of this conjecture to so-called S-duality by Sen [4] has attracted much attention. The picture that emerges from Sen's work and the evidence to support his conjecture [5] is that $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with abelian unbroken gauge group H contains electric, dyonic and magnetic charge sectors, and that these are mapped into each other by the duality group $SL(2, \mathbf{Z})$.

In this paper we continue our investigation of charged excitations in Yang-Mills-Higgs theory with non-abelian unbroken gauge symmetry, begun in [6]. Our goal is to understand the structure of the charge sectors and to formulate S-duality in this setting. Precisely we consider Yang-Mills theory on (3+1)-dimensional Minkowski space, with gauge group G and complex coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$ (combining the coupling constant e with the θ-angle of the theory). We will mostly consider the $\mathcal{N} = 4$ supersymmetric version of the theory here. Although we will not perform any explicitly supersymmetric computations, embedding our arguments in the $\mathcal{N} = 4$ supersymmetric setting allows us to make certain quantitative statements. Since the $\mathcal{N} = 4$ theory has a vanishing β-function we are in particular able to refer to a scale independent coupling constant. We should stress, however, that we expect the qualitative aspects of the picture we are going to present to remain valid even in the non-supersymmetric situation.

To formulate the problems we have to address as clearly as possible, we briefly review the abelian situation. All the relevant features are present in the simplest model, that of $G = SU(2)$ broken to $H = U(1)$. In that theory the electric charge of an excitation is given by a single integer N, the label of an UIR of $U(1)$. The magnetic charge also takes integer values, which we denote by K. While the mathematical status of that integer is topological (it is the degree of the Higgs field at spatial infinity) it can fruitfully be interpreted as a representation label of a magnetic $U(1)$. To distinguish the electric from the (hypothetical) magnetic $U(1)$ we write $\tilde{U}(1)$ for the latter. A general dyonic sector may thus be characterised by a pair of integers (K, N). The important point here is that the fusion rule for dyonic sectors

$$(K_1, N_1) \otimes (K_2, N_2) = (K_1 + K_2, N_1 + N_2) \tag{1.1}$$

is indeed the Clebsch-Gordan series for representations of $\tilde{U}(1) \times U(1)$. While this remark is a triviality in the abelian case, we will see that the fusion properties of dyonic sectors impose severe constraints on the group-theoretical interpretation of magnetic charges in the non-abelian context.

The duality group in this case is $SL(2, \mathbf{Z})$, which acts naturally on the pair (K, N). Thus the element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \tag{1.2}$$

maps the sector (K, N) onto the sector $(K, N)M^{-1} = (dK - cN, -bK + aN)$ while transforming simultaneously the coupling τ via a modular transformation to $(a\tau + b)/(c\tau + d)$. In particular the element

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.3}$$

implements the Montonen-Olive electric-magnetic (and weak-strong coupling) duality [3], and the element

$$T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{1.4}$$

implements the Witten effect (the 2π-shift in the θ-angle) [7].

Finally we emphasise that the $SL(2, \mathbf{Z})$ action on the sectors respects the fusion rule (1.1). In fact one can invert the logic and ask which permutation of the sectors (K, N) is an automorphism of the fusion rules. Any permutation Π of the sectors can be expressed in terms of an invertible map $\pi : \mathbf{Z}^2 \to \mathbf{Z}^2$ of the integer labels (K, N). The requirement

$$\Pi((K_1, N_1)) \otimes \Pi((K_2, N_2)) = \Pi((K_1 + K_2, N_1 + N_2)) \tag{1.5}$$

means that the associated map π is linear. Since invertible linear maps $\mathbf{Z}^2 \to \mathbf{Z}^2$ constitute the group $SL(2,\mathbf{Z})$ one could define the S-duality group also as the automorphism group of the fusion ring. This point of view will play an important role in our discussion.

There are two principal problems which one encounters when trying to generalise the above story to the non-abelian regime. The first concerns the identification of the magnetic group. While the interpretation of the magnetic charge lattice as the weight lattice of the magnetic group goes back to [2], the precise identification of monopole solutions one finds classically with the UIR's of the conjectured magnetic group has so far not been achieved, despite various efforts [8].

The second problem concerns the dyonic sectors of the theory. It was first noticed by Nelson and Manohar [9], further elaborated by Horvathy and Rawnsley [10] and more recently by us [6] that in the presence of non-abelian magnetic charge only that part of the unbroken (electric) group has a globally defined action on a classical configuration which commutes with the magnetic charge. Thus, unlike in the abelian case, we cannot expect to label dyonic sectors by UIR's of the product of the electric and the magnetic group. Rather we require a labelling which accounts for the interplay between magnetic and electric charges.

We offer solutions to both these problems here. The essential input which allows us to overcome the first of the above problems is the interpretation of the classical monopole solutions as coherent states of the magnetic group. Putting this together with the results of our earlier paper [6] we present a consistent labelling of the magnetic, dyonic and electric sectors of the theory. Purely electric sectors are labelled by UIR's of the electric group H and purely magnetic sectors by UIR's of the magnetic group $\tilde{H}$, but the important point is that electric and magnetic symmetry are never simultaneously manifest: one can at most implement subgroups of H and $\tilde{H}$ which, in a sense to be specified in this paper, commute with each other.

The fusion rules of the sectors are considerably more intricate than in the abelian case and depend on the coupling regime; they were discussed in [6] for the case of weak electric coupling $e \ll 1$. Here we are able to use our evidence for the dual group $\tilde{H}$ to understand the fusion rules also in the strong electric coupling regime $e \gg 1$. Our picture includes a truly non-abelian implementation of electric-magnetic duality. Moreover we find that, if we restrict attention to coherent states (both magnetically and electrically), the fusion rules simplify and S-duality can again be formulated as an $SL(2,\mathbf{Z})$-action on charge sectors which leaves the fusion rules invariant.

In this paper we shall explain our ideas in Yang-Mills theory with gauge group $SU(3)$ broken to $U(2)$. This is the simplest model which displays all the phenomena we want to discuss. Our main reference throughout is the paper [6], where the semi-classical properties of monopoles in this model were discussed in detail. Nonetheless the present paper can be read independently; whenever results from [6] are used we have stated them carefully.

2. Classical monopoles as coherent states of the magnetic group

2.1. Classical Monopoles

The bosonic fields of $\mathcal{N}=4$ supersymmetric Yang-Mills theory with coupling constant e are a connection and an adjoint Higgs field. A classical (static) monopole solution is a pair (A_i, Φ) of such a connection (we work in the temporal gauge $A_0 = 0$) and a Higgs field on $\mathbf{R}^3$ satisfying the Bogomol'nyi equation:

$$D_i \Phi = B_i \tag{2.1}$$

as well as certain boundary conditions. Here $D_i = \partial_i + eA_i$ is the covariant derivative and B_i is the non-abelian magnetic field

$$B_i = \frac{1}{2}\epsilon_{ijk}\left(\partial_j A_k - \partial_k A_j + e[A_j, A_k]\right). \tag{2.2}$$

For details about the boundary conditions and the notational conventions concerning the gauge group $SU(3)$ we refer the reader to [6]. Here we note only those boundary conditions which concern the symmetry breaking and the magnetic charge. The symmetry breaking scale is set by

$$-\mathrm{tr}\,\Phi^2 \to \frac{1}{2}v^2 \quad \text{for } r \to \infty. \tag{2.3}$$

Further we demand that the Higgs field has the following form along the positive z-axis:

$$\Phi(0,0,z) = \Phi_0 - \frac{G_0}{4\pi z} + \mathcal{O}(\frac{1}{z^2}), \tag{2.4}$$

where Φ_0 and G_0 are constant elements of the Lie algebra $su(3)$. The former determines the symmetry breaking pattern and may be chosen to lie in the Cartan subalgebra. For the minimal symmetry breaking case we are interested in we require Φ_0 to have one repeated eigenvalue, so we take

$$\Phi_0 = \frac{iv}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{2.5}$$

Then the generators of the unbroken $U(2)$ symmetry have the following form at $z = +\infty$:

$$\begin{aligned} I_1 &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; I_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; I_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ Y &= \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned} \tag{2.6}$$

The constant Lie algebra element G_0 is the magnetic charge of the configuration and has to satisfy the generalised Dirac quantisation condition [1], [2]:

$$\exp(eG_0) = 1. \tag{2.7}$$

The magnetic charge G_0 may also be rotated into the Cartan subalgebra, and then the Dirac condition forces it to lie on a certain lattice, the dual root lattice of $su(3)$. However, in the case of minimal symmetry breaking it is not natural to require G_0 to lie in Cartan subalgebra. A better way to characterise the magnetic charge is to consider the orbit of G_0 under the action of the gauge group $U(2)$, acting in the base point $(0, 0, \infty)$. As shown in [6] these orbits are either trivial, in which case they are points of quantised "height" in the Lie algebra $su(3)$, or two-spheres of quantised radius and "height" in $su(3)$. These orbits are thus characterised by two numbers K and k, where K (the "height") is an integer and k (the radius) is a non-negative half-odd integer if K is odd and a non-negative integer if K is even; see Fig. 1. Explicitly, each non-trivial orbit can be parametrised by spherical coordinates $(\alpha, \beta) \in [0, 2\pi) \times [0, \pi)$ so that an element of the orbit labelled by (K, k) can be written as

$$G_0(\alpha, \beta) = \frac{4K\pi i}{e}\left(\frac{3}{4}Y\right) + \frac{4\pi i}{e}\mathbf{k}\cdot\mathbf{I}, \tag{2.8}$$

where the vector $\mathbf{k} = (k_1, k_2, k_3)$ has length $|\mathbf{k}| = k$ and the direction $\hat{\mathbf{k}}$ parametrised by (α, β):

$$\hat{\mathbf{k}} = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta). \tag{2.9}$$

We emphasise that the coordinates (α, β) stem from the action of the unbroken gauge group $U(2)$ at one point only. As mentioned in Sect. 1, this action cannot be extended to a global action in the presence of non-abelian magnetic charge. Clarifying the physical interpretation of the magnetic orbits and their coordinates is one of the goals of this section.

To sum up, the allowed magnetic charges of a monopole can be written as a pair $(K, \mathbf{k})$ consisting of an integer K and a vector $\mathbf{k}$ of quantised length k such that $K + 2k \in 2\mathbf{Z}$. As explained in [6] it follows from the results of [11] that for solutions of the Bogomol'nyi equations the charges necessarily lie inside the cone $k \leq |K|/2$.

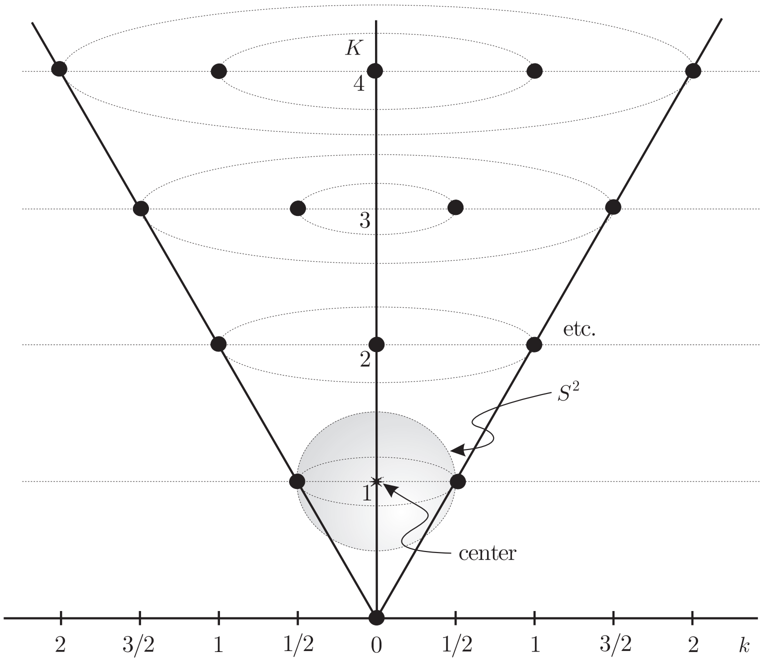

Fig. 1: *The magnetic orbits classifying $SU(3)$ monopoles with minimal symmetry breaking*

We are now able to phrase in more precise terms the first of the two principal problems outlined in the introduction, that of identifying the magnetic group. By definition that group should have UIR's which classify the magnetic monopoles found in the theory. The challenge is thus to identify the classical set of monopoles whose charges lie on an orbit with labels K and k with the UIR of some group. We shall now show how this can be achieved and how a magnetic group can be defined. The key step is the interpretation of the classical monopole charges as labels of coherent states. In anticipation of this result we introduce the notation

$$|K; k, \hat{\mathbf{k}}\rangle \tag{2.10}$$

for monopole charges, where we have separately notated the length and the direction of the non-abelian charge $\mathbf{k}$.

2.2. Coherent states revisited

The best-known coherent states are related to the Heisenberg group, but various generalisations have been studied in the literature. For us the concept of coherent states for general compact Lie groups introduced in [12] is particularly relevant. In this paper we mainly need coherent states of $SU(2)$ which are also much discussed in the literature, usually under the names "Bloch states" or "spin coherent states". For a guide to the vast literature on the subject we refer the reader to the collection of reprints [13]. There exist various different conventions for defining coherent states, mostly to do with choosing a "fiducial state". We consider coherent states defined in terms of a highest weight state, but in order to avoid confusion we will carefully state our conventions. Thus we introduce Euler angles $(\alpha,\beta,\tilde{\gamma}) \in [0,2\pi)\times[0,\pi)\times[0,4\pi)$ by writing an $SU(2)$ matrix P in terms of the Pauli matrices τ_1,τ_2 and τ_3 as

$$P(\alpha,\beta,\tilde{\gamma}) = e^{-\frac{i}{2}\alpha\tau_3}e^{-\frac{i}{2}\beta\tau_2}e^{-\frac{i}{2}\tilde{\gamma}\tau_3}. \tag{2.11}$$

The coherent state $|k,\hat{\mathbf{k}}\rangle$ in a spin k representation V_k of $SU(2)$ with basis $\{|k,m\rangle\}$ ($m = -k,-k+1,...,k-1,k$) is defined in terms of the highest weight state $|k,k\rangle$ as

$$|k,\hat{\mathbf{k}}\rangle = D^k(P)|k,k\rangle = \sum_{m=-k}^{k} D^k_{mk}(P)|k,m\rangle, \tag{2.12}$$

where $D^k(P)$ is the representation of P in V_k and

$$D^k_{ms}(P) = \langle k,m|D^k(P)|k,s\rangle = e^{-im\alpha}d^k_{ms}(\beta)e^{-is\tilde{\gamma}} \tag{2.13}$$

are the Wigner functions in the conventions of [14]. The coherent states are labelled by the total spin k and the unit vector $\hat{\mathbf{k}}$, which is the image of $P \in SU(2) \sim S^3$ under the Hopf projection

$$\hat{\mathbf{k}} = \pi_{\mathrm{Hopf}}(P), \tag{2.14}$$

where $\hat{\mathbf{k}}$ is given in terms of the Euler angles (α,β) as in (2.9). Coherent states only depend on the angle $\tilde{\gamma}$ via a phase factor and it is conventional in the discussion of coherent states to choose $\tilde{\gamma} = 0$.

The expectation value of the spin operator $\mathbf{J} = (J_1, J_2, J_3)$ has the following simple form for coherent states:

$$\langle k, \hat{\mathbf{k}} | \mathbf{J} | k, \hat{\mathbf{k}} \rangle = k\hat{\mathbf{k}}. \tag{2.15}$$

In particular the length of the expectation value of the spin operator is thus equal to the total spin k for a coherent state. This property can also be used as the defining property of coherent states. This approach gives a better conceptual understanding and we want to adopt it here although it does not seem to be standard in the literature. To appreciate the significance of the definition we are going to give it is useful to recall some simple facts about $SU(2)$ representations.

Fact 1: Let $|\psi\rangle$ be a vector in the carrier space V_k of the spin k representation of $SU(2)$. Then the expectation value $\mathbf{j}_\psi = \langle \psi | \mathbf{J} | \psi \rangle$ of the spin operator has length at most k: $|\mathbf{j}_\psi| \leq k$.

To see this use the relations

$$\begin{aligned} J_+^{\hat{k}} J_-^{\hat{k}} &= \mathbf{J}^2 - (\hat{\mathbf{k}} \cdot \mathbf{J})^2 + \hat{\mathbf{k}} \cdot \mathbf{J} \\ J_-^{\hat{k}} J_+^{\hat{k}} &= \mathbf{J}^2 - (\hat{\mathbf{k}} \cdot \mathbf{J})^2 - \hat{\mathbf{k}} \cdot \mathbf{J} \end{aligned} \tag{2.16}$$

for an arbitrary quantisation axis $\hat{\mathbf{k}}$ and operators $J_+^{\hat{k}}$ and $J_-^{\hat{k}}$ which act as raising and lowering operators in the spectrum of $\hat{\mathbf{k}} \cdot \mathbf{J}$. Computing the expectation value of both sides for an arbitrary state $|\psi\rangle \in V_k$ and using the positivity of the resulting LHS one deduces

$$k(k+1) \geq |\hat{\mathbf{k}} \cdot \mathbf{j}_\psi| (|\hat{\mathbf{k}} \cdot \mathbf{j}_\psi| + 1) \Rightarrow k \geq |\hat{\mathbf{k}} \cdot \mathbf{j}_\psi|. \tag{2.17}$$

Since this holds for all $\hat{\mathbf{k}} \in S^2$ the claim follows.

An immediate corollary is the

Fact 2: For all $|\psi\rangle \in V_k$ the square of the "uncertainty" of the spin vector $\sigma_\psi^2(\mathbf{J}) = \sum_{a=1}^{3} \langle \psi | J_a^2 | \psi \rangle - \langle \psi | J_a | \psi \rangle^2$ is bounded below by k.

It is easy to check that the bounds on the length of $\mathbf{j}_\psi$ and the variance $\sigma_\psi(\mathbf{J})$ are attained for the coherent states (2.12). Conversely we can define coherent states of $SU(2)$ to be those states $|\psi\rangle$ in the carrier space V_k of the spin k representation for which the length of expected spin vector $\mathbf{j}_\psi$ is maximal $(= k)$ and the variance $\sigma_\psi(\mathbf{J})$ is minimal $(= \sqrt{k})$. A coherent state $|\psi\rangle \in V_k$ can thus be characterised uniquely by the direction $\hat{\mathbf{k}}$ of $\mathbf{j}_\psi$. In this approach, the equation (2.15) is the defining equation for the coherent state $|k, \hat{\mathbf{k}}\rangle$.

Coherent states are over-complete. In particular the inner product $\langle k, \hat{\mathbf{k}}' | k, \hat{\mathbf{k}} \rangle$ of two coherent states in the spin k representation only vanishes between the coherent states associated with antipodal points, i.e. if $\hat{\mathbf{k}}' = -\hat{\mathbf{k}}$. Finally we note the following decomposition of the identity in terms of coherent states in the carrier space V_k:

$$\mathrm{Id} = \frac{2k+1}{4\pi} \int \sin\beta d\beta d\alpha \; |k, \hat{\mathbf{k}}\rangle\langle k, \hat{\mathbf{k}}|. \tag{2.18}$$

2.3. The magnetic group

To make contact between magnetic charges and coherent states we need to include the $U(1)$-factor of $U(2) = (U(1) \times SU(2))/\mathbf{Z}_2$ in our picture. All $U(1)$ representations are one-dimensional and a normalised basis state $|K\rangle$ in the charge K representation of $U(1)$ is automatically coherent. We denote the carrier space of an UIR of $U(2)$ by $V_{K,k}$, where K is the $U(1)$ charge and k the $SU(2)$ spin and we have the the constraint $K + 2k \in 2\mathbf{Z}$ in order to respect the $\mathbf{Z}_2$ identification. Then a coherent state in $V_{K,k}$ is of the form

$$|K; k, \hat{\mathbf{k}}\rangle = |K\rangle \otimes |k, \hat{\mathbf{k}}\rangle. \tag{2.19}$$

It is clear from our notation that there is a natural bijection between monopole charges (2.10) and coherent states (2.19) of $U(2)$. Here we propose to identify the two. Thus we interpret the charges (K, k) characterising the magnetic orbits as labels of UIR's of a magnetic copy of $U(2)$. To distinguish this $U(2)$ from the unbroken electric group $U(2)$ we denote it by $\tilde{U}(2)$.

If our interpretation of the classical monopole solutions found in the weak electric coupling regime as coherent states of the magnetic group $\tilde{U}(2)$ is correct, then there is presumably some fundamental principle at work which forces the fundamental magnetic degrees of freedom to form coherent states in the weak electric coupling regime. Since we do not have a formulation of Yang-Mills theory where both electric and magnetic symmetry are manifest we are not able to identify such a principle. In the absence of a deductive proof of our proposal we will therefore assess its value by studying some of its consequences.

One immediate implication of the proposal is that there is a natural mapping from the continuous magnetic orbits found in the classification of classical monopoles to finite-dimensional representation spaces of the magnetic group. Mathematically this mapping is precisely the one one would obtain according to the geometric quantisation prescription, but our physical interpretation is different. Whereas geometric quantisation would suggest that the magnetic orbits are classical phase spaces with quantised volume, we interpret the continuous parameters of the magnetic orbits as labels of the over-complete set of coherent states. By inverting the relation (2.12) we are thus in particular able to write down the basis of the carrier space $V_{K,k}$ in terms of the coherent states, thus solving the long-standing problem of isolating the fundamental magnetic degrees of freedom in the weak electric coupling regime:

$$|K;k,m\rangle = \int \sin\beta d\beta d\alpha\ D^{k*}_{mk}(P)|K;k,\hat{\mathbf{k}}\rangle. \tag{2.20}$$

In particular we deduce that an element $(e^{i\chi},g) \in \tilde{U}(2)$ of the magnetic group acts on $V_{K,k}$ according to

$$D^{K,k}(e^{i\chi},g)|K;k,\hat{\mathbf{k}}\rangle = e^{iK\chi}e^{ik\delta}|K;k,\mathcal{G}\hat{\mathbf{k}}\rangle \tag{2.21}$$

where $\mathcal{G}$ is the $SO(3)$ matrix associated to the $SU(2)$ matrix g. The angle δ depends on g and $\hat{\mathbf{k}}$ in a way which is quite complicated and not very illuminating, see e.g. [14]. Note that if we start with a reference state $|K;k,(0,0,1)\rangle$ the vector $\hat{\mathbf{k}} = \mathcal{G}(0,0,1)^t$ sweeps out the magnetic orbits shown in Fig. 1 under the action of the magnetic group. Thus, although we generated the coordinates (α,β) in (2.9) originally by acting with the unbroken gauge group $U(2)$ in one point, we now find that we should interpret them as stemming from the action of the magnetic group $\tilde{U}(2)$. However, the magnetic group action not only rotates the charge vector $\hat{\mathbf{k}}$ but also generates the K- and k-dependent phase factors in (2.21).

An interesting check of our proposal is to see whether it allows us to reproduce (and perhaps better understand) the fusion rule for magnetic charges discussed in [6]. According to that rule monopoles with magnetic charges $(K_1,\mathbf{k}_1)$ and $(K_2,\mathbf{k}_2)$ can be combined only if the non-abelian components are either parallel or anti-parallel. Then the charges are added like vectors and the combined charge automatically satisfies the Dirac quantisation condition if the individual charges do. In the language of coherent states the natural way of "fusing" two states is to compute their tensor product. Here there is no reason to impose any condition on the states to be multiplied. The general formula is complicated:

$$\begin{aligned}|K_1;k_1,\hat{\mathbf{k}}_1\rangle \otimes |K_2;k_2,\hat{\mathbf{k}}_2\rangle = \\ \sum_{k=|k_1-k_2|}^{k_1+k_2}\sum_{m=-k}^{k}\sum_{s=-k_2}^{k_2} D^{k_2}_{sk_2}(P_1^{-1}P_2)C^{k,s+k_1}_{k_1k_1,k_2s}D^{k}_{m,s+k_1}(P_1)|K_1+K_2;k,m\rangle,\end{aligned} \tag{2.22}$$

where P_1 and P_2 are $SU(2)$ matrices whose images under the Hopf projection (2.14) are $\hat{\mathbf{k}}_1$ and $\hat{\mathbf{k}}_2$ respectively, and $C^{k,s+k_1}_{k_1k_1,k_2s}$ is an $SU(2)$ Clebsch-Gordan coefficient. After expanding

$|K_1 + K_2; k, m\rangle$ according to (2.20) the above formula shows that the product of two coherent states of spins k_1 and k_2 is a superposition of coherent states with spins between $|k_1 - k_2|$ and $k_1 + k_2$. Rather remarkably one obtains in this way a closed multiplication rule for "vectors of quantised length".

If it is true that only coherent states of the magnetic group show up in the weak electric coupling regime we should be able to reproduce our classical fusion rule by projecting out the coherent states in the tensor product. More precisely we project the RHS of (2.22) onto each of the carrier spaces V_k, $|k_1 - k_2| \leq k \leq k_1 + k_2$, and check whether the projected state is coherent. In general this is not the case, but it does happen when $\hat{\mathbf{k}}_1$ and $\hat{\mathbf{k}}_2$ are parallel or anti-parallel. In the former case, when $\hat{\mathbf{k}}_1$ and $\hat{\mathbf{k}}_2$ are equal to, say, $\hat{\mathbf{k}}$, the product state is coherent:

$$|K_1; k_1, \hat{\mathbf{k}}\rangle \otimes |K_2; k_2, \hat{\mathbf{k}}\rangle = |K_1 + K_2; k_1 + k_2, \hat{\mathbf{k}}\rangle. \tag{2.23}$$

In the latter case, the product state contains a coherent state in the carrier space V_k with the lowest possible spin $k = |k_1 - k_2|$:

$$|K_1; k_1, \hat{\mathbf{k}}\rangle \otimes |K_2; k_2, -\hat{\mathbf{k}}\rangle = |K_1 + K_2; k_1 - k_2, \hat{\mathbf{k}}\rangle + \text{incoherent states}, \tag{2.24}$$

where we assumed without loss of generality that $k_1 \geq k_2$. If $|k_1 - k_2| = 0$ the product state should be interpreted as the unique singlet state. Thus we indeed recover the classical selection rule on monopole charges which may be multiplied and also reproduce the classical results in the cases where the classical multiplication is allowed.

2.4. Non-abelian electric-magnetic duality

The identification of magnetic monopoles with states in UIR's of the magnetic group is a necessary condition for the validity of any formulation of non-abelian electric-magnetic duality. Having achieved this identification we can now go further and extend the Montonen-Olive electric-magnetic duality conjecture to the case of non-abelian unbroken gauge symmetry. Specifically in the theory we are considering here we propose that the physics of purely electric particles in the UIR (N, j) of $U(2)$ at coupling e is the same as that of purely magnetic particles in the UIR of $\tilde{U}(2)$ with labels $(K, k) = (-N, j)$ at coupling $4\pi/e$. The coupling constant e is that of the electric formulation of the theory, so $4\pi/e$ should be thought of as the coupling constant of a dual or magnetic formulation of the theory. In particular we therefore refer to the regime $e \ll 1$ as the weak electric coupling regime and to the regime $e \gg 1$ as the weak magnetic coupling regime.

One immediate and interesting corollary of the non-abelian electric-magnetic duality conjecture concerns the coherency requirement on electric and magnetic degrees of freedom. Since no such requirement applies to electric states in the weak electric coupling regime we deduce that magnetic degrees of freedom are not necessarily coherent in the weak magnetic coupling regime. Conversely it follows from our coherency postulate for magnetic states in the weak electric coupling regime that in the weak magnetic (=strong electric) coupling regime only electric coherent states are allowed. In particular they would therefore have to obey the selection rule for tensor products, discussed in its magnetic version above: only electric coherent states with parallel or anti-parallel charge directions may be combined in a tensor product. This last selection rule for electric states is reminiscent of a discussion by Corrigan of point-sources for static Yang-Mills fields. In [15] it was pointed out that only point sources with non-abelian classical charges which lie on the same Weyl-orbit of the gauge group can be combined to produce a static field. For the group $U(2)$ discussed here two charges lie on the same Weyl orbit precisely if they have parallel or anti-parallel non-abelian charge directions $\hat{\mathbf{k}}$.

Non-abelian electric magnetic duality is also crucial in getting a complete picture of the possible charge sectors and their fusion properties in Yang-Mills-Higgs theory with non-abelian unbroken gauge group. Since this is one of the main concerns of this paper we have devoted a separate section to it.

3. Charge sectors and fusion rules

In this paper we use the term "fusion" in a general sense to refer to the process of combining different charge sectors of the theory. In particle theory different charge sectors are usually in one-to-one correspondence with UIR's of some group, and the answer to the fusion problem is then given by the Clebsch-Gordan series of that group. However, it is well-known that in particular in two-dimensional theories fusion properties are sometimes dictated by the representation ring of other algebraic structures, such as quantum groups. Here we will encounter yet a different situation: we are not able to sum up the fusion properties in the representation ring of a single algebraic object. Instead the fusion properties depend on the coupling constant e, with different groups classifying the sectors and organising the fusion behaviour in the weak electric coupling regime and the weak magnetic coupling regime. Only certain states which are present at all values of the coupling - the coherent states - obey universal fusion rules.

We begin our classification of the charge sectors with non-abelian electric sectors, and initially consider the weak electric coupling regime. Non-abelian electric sectors are defined by the absence of non-abelian magnetic charge k. States in this regime transform under the electric group $U(2)$ and can be grouped into UIR's of that group. Such UIR's are labelled by an integer N and a positive half-integer spin j satisfying $N+2j \in 2\mathbf{Z}$. A basis of states for this sector is furnished by the tensor product $|N;j,m\rangle = |N\rangle \otimes |j,m\rangle$ of the $U(1)$ state $|N\rangle$ with the customary basis states $\{|j,m\rangle | m=-j,-j+1,...,j-1,j\}$ of the spin j representation of $SU(2)$. If the abelian magnetic charge K is zero the sector is purely electric and contains the familiar perturbative massless and massive states. If the magnetic charge has the form $(K \neq 0, k=0)$ (i.e. it lies on one of the magnetic orbits on the vertical axis in Fig. 1.) the integer K is necessarily even and the excitations, studied in [6], are then dyonic. Such dyonic sectors are thus labelled by the triplet (K,N,j) and for later use we introduce basis states via

$$|K;N;j,m\rangle = |K\rangle \otimes |N;j,m\rangle. \tag{3.1}$$

The fusion rules of the non-abelian electric sectors are dictated by the representation ring of the group $\tilde{U}(1) \times U(2)$: the abelian magnetic charges simply add and the electric charges combine according to the familiar Clebsch-Gordan series of $U(2)$.

To extend our understanding of non-abelian electric charge sectors to the strong electric coupling regime we use the corollary of non-abelian electric-magnetic duality noted at the end of the previous section: in the strong electric coupling regime only coherent states of the electric group $U(2)$ are physical. Considering without loss of generality the case of $K=0$, the coherent electric states are superpositions of the basis vectors $|N\rangle \otimes |j,m\rangle$ of the $U(2)$ UIR (N,j) introduced above:

$$|N;j,\hat{\mathbf{k}}\rangle = |N\rangle \otimes \sum_{m=-j}^{j} D^j_{mj}(Q)|j,m\rangle. \tag{3.2}$$

The $SU(2)$ matrix Q is parametrised by Euler angles (α,β,γ) and the direction of $\hat{\mathbf{k}}$ is given in terms of (α,β) as in the formula (2.9) for the magnetic charge direction. The use of the same angular coordinates to parametrise both magnetic and electric charge directions is no accident but a manifestation of the deep result that magnetic and electric symmetry are never simultaneously realised. Thus it is possible that, depending on the charge sector, the same coordinates may have an electric or a magnetic interpretation. In [6] we saw for example that in a fusion process magnetic parameters (generated by the action of the magnetic group) acquire an electric interpretation. We will return to such fusion processes after we have completed the classification of charge sectors. Here we note that within the non-abelian electric charge sector at strong electric coupling the fusion rules are the same as those in the non-abelian magnetic charge sectors at strong magnetic coupling: only states with parallel or anti-parallel charge directions $\hat{\mathbf{k}}$ may be multiplied, and the result is

$$\begin{aligned} |N_1; j_1, \hat{\mathbf{k}}\rangle \otimes |N_2; j_2, \hat{\mathbf{k}}\rangle &= |N_1 + N_2; j_1 + j_2, \hat{\mathbf{k}}\rangle \\ |N_1; j_1, \hat{\mathbf{k}}\rangle \otimes |N_2; j_2, -\hat{\mathbf{k}}\rangle &= |N_1 + N_2; j_1 - j_2, \hat{\mathbf{k}}\rangle + \text{incoherent states.} \end{aligned} \tag{3.3}$$

Next consider non-abelian magnetic charge sectors, defined by the absence of non-abelian electric charge, i.e. $j = 0$. At weak electric coupling non-abelian magnetic degrees of freedom necessarily form coherent states (2.19) and obey the fusion rules (2.23) and (2.24). By duality the requirement of coherency no longer applies in the weak magnetic coupling regime. Magnetic states are then arbitrary elements of UIR's of the magnetic group $\tilde{U}(2)$. Such UIR's are labelled as before by a pair (K, k) of an integer K and a positive half-integer k satisfying $K + 2k \in 2\mathbf{Z}$. The abelian electric charge N may be zero, in which case the sector is purely magnetic. If the electric charge N is even but non-zero we have dyonic sectors labelled by the triplet (K, k, N), characterising UIR's of $\tilde{U}(2) \times U(1)$. The fusion rules of states in these sectors are given by the Clebsch-Gordan coefficients of that group. Physical states in these sectors are tensor products of the non-abelian magnetic states and abelian electric states. At weak electric coupling they are magnetically coherent (2.19):

$$|K; k, \hat{\mathbf{k}}; N\rangle = |K; k, \hat{\mathbf{k}}\rangle \otimes |N\rangle. \tag{3.4}$$

At weak magnetic coupling a basis can be written in terms of the magnetic states (2.20):

$$|K; k, m; N\rangle = |K; k, m\rangle \otimes |N\rangle, \tag{3.5}$$

and arbitrary linear combinations of these basis states are allowed.

Finally we turn to non-abelian dyonic sectors, described in great detail in [6] in the weak electric coupling regime. There the non-abelian magnetic charge of a coherent magnetic state obstructs the implementation of the electric group $U(2)$. More precisely, on a monopole of charge $(K, \mathbf{k} \neq 0)$ only that subgroup of $U(2)$ can be implemented which leaves the magnetic charge invariant. In this case this is a maximal torus $T^2(\hat{\mathbf{k}})$ of $U(2)$ characterised by the unit vector $\hat{\mathbf{k}}$. For a physical interpretation of the corresponding charges it is useful to separate the diagonal $U(1)$ subgroup of $U(2)$ and write the maximal torus as $T^2(\hat{\mathbf{k}}) = (U(1) \times T^1(\hat{\mathbf{k}}))/\mathbf{Z}_2$, where $T^1(\hat{\mathbf{k}})$ is the torus subgroup in $SU(2)$ generated by $(\hat{k}_1\tau_1 + \hat{k}_2\tau_2 + \hat{k}_3\tau_3)$ (the τ_i are the Pauli matrices) and $\mathbf{Z}_2$ is the group $\{1, -1\}$. Dyonic quantum states are characterised by giving the underlying monopole state as in (2.10) and then specifying the $(U(1) \times T^1(\hat{\mathbf{k}}))/\mathbf{Z}_2$ charges (N, n), the former being an integer and the second a half-integer such that $N + 2n$ is even. Such dyonic states can thus be written as

$$|K, k; N, n; \hat{\mathbf{k}}\rangle. \tag{3.6}$$

By duality we expect the following description of dyons to hold in the weak magnetic coupling regime. Now electric states are necessarily coherent, and if they carry non-abelian electric charge it will obstruct the implementation of the magnetic group $\tilde{U}(2)$. More precisely on an electric coherent state $|N; j, \hat{\mathbf{k}}\rangle$ only that subgroup of $\tilde{U}(2)$ can be implemented which leaves $\hat{\mathbf{k}}$ invariant. This is the maximal torus $\tilde{T}^2(\hat{\mathbf{k}})$ which, in analogy with the electric torus $T^2(\hat{\mathbf{k}})$, we rewrite as $\tilde{T}^2(\hat{\mathbf{k}}) = (\tilde{U}(1) \times \tilde{T}^1(\hat{\mathbf{k}}))/\mathbf{Z}_2$. Denoting the eigenvalues of the $\tilde{U}(1)$ and $\tilde{T}^1(\hat{\mathbf{k}})$ generators respectively by K and k, a dyonic state in the weak magnetic coupling regime can be written as $|K, k; N, j; \hat{\mathbf{k}}\rangle$, which has precisely the same structure as the weak electric coupling state (3.6). This similarity suggests that we could think of dyonic states more symmetrically as charged with respect to the subgroup

$$\tilde{T}^2(\hat{\mathbf{k}}) \times T^2(\hat{\mathbf{k}}) \subset \tilde{U}(2) \times U(2). \tag{3.7}$$

Our discussion of the charge sectors can be summed up in the following table:

Sector type	Symmetry group	States at $e \ll 1$	States at $e \gg 1$
electric	$\tilde{U}(1) \times U(2)$	$\|K; N; j, m\rangle$	$\|K; N; j, \hat{\mathbf{k}}\rangle$
dyonic	$\tilde{T}^2(\hat{\mathbf{k}}) \times T^2(\hat{\mathbf{k}})$	$\|K, k; N, n; \hat{\mathbf{k}}\rangle$	$\|K, k; N, n; \hat{\mathbf{k}}\rangle$
magnetic	$\tilde{U}(2) \times U(1)$	$\|K; k, \hat{\mathbf{k}}; N\rangle$	$\|K; k, m; N\rangle$

Table 1: *Non-abelian charge sectors and their states*

Remarkably the groups listed in table 1 are all subgroups of $\tilde{U}(2) \times U(2)$ with the property that the two factors centralise each other. This is the basis of the general statement made in the introduction that quantum states in Yang-Mills theory with non-abelian unbroken gauge group H can at most be charged with respect to subgroups of $\tilde{H}$ and H which centralise each other (when H and $\tilde{H}$ are not isomorphic - such as in the case of non-simply laced H - the centralising property should be defined in the adjoint representation). It is satisfying that one can sum up the intricate interplay between electric and magnetic charges in this neat, general statement. In particular this result suggests that the following union of commuting pairs in $\tilde{U}(2) \times U(2)$

$$U(2)_{\rm com} = \Big(\tilde{U}(2) \times U(1)\Big) \cup \left(\bigcup_{\hat{\mathbf{k}} \in S^2} \tilde{T}^2(\hat{\mathbf{k}}) \times T^2(\hat{\mathbf{k}}) \right) \cup \Big(\tilde{U}(1) \times U(2)\Big) \tag{3.8}$$

plays a central role in understanding the charge sectors of Yang-Mills-Higgs theory with unbroken gauge group $U(2)$. We have used the notation $U(2)_{\rm com}$ because this set can also be defined as the set of commuting pairs of elements in $\tilde{U}(2) \times U(2)$:

$$U(2)_{\rm com} = \{(g,h) \in \tilde{U}(2) \times U(2) | gh = hg\}. \tag{3.9}$$

However, while this set has many interesting properties, it does not appear to be endowed with a natural algebraic structure. In particular it is not closed under multiplication. The product of two elements in $U(2)_{\rm com}$ only belongs to $U(2)_{\rm com}$ if the two elements both belong to one of the groups in the decomposition (3.8). As we have seen, the Clebsch-Gordan series of those groups tells us how to combine two sectors of the same type (for example two non-abelian electric sectors), but the structure of $U(2)_{\rm com}$ does not tell us how to combine a non-abelian electric state with a dyonic state, charged with respect to $\tilde{T}^2(\hat{\mathbf{k}}) \times T^2(\hat{\mathbf{k}})$ for some $\hat{\mathbf{k}}$. To answer that question we need to combine results from our earlier paper [6] with non-abelian electric-magnetic duality.

The key result of the paper [6] is that in the weak electric coupling regime electric, magnetic, and dyonic states can all be interpreted as carrying representations of the semi-direct product group $U(2) \ltimes \mathbf{R}^4$. Here the group $\mathbf{R}^4$ should be thought of as a "magnetic translation group": elements of the dual $(\mathbf{R}^*)^4$ are physically interpreted as magnetic charges. As vector spaces $\mathbf{R}^4$ and the Lie algebra of $U(2)$ are isomorphic, and $U(2)$ acts on an element of $\mathbf{R}^4$ by conjugation of the associated element of the Lie algebra of $U(2)$. We remind the reader that representations of semi-direct products (like the Poincaré group or the Euclidean group) are characterised by orbits and centraliser representations. In the case of $U(2) \ltimes \mathbf{R}^4$ the relevant orbits are those of $U(2)$ acting on $(\mathbf{R}^*)^4$. If the orbit is trivial (one of the points on the central axis of Fig. 1) representations are purely electric $U(2)$ representations. If the orbit is non-trivial (one of the two-spheres in Fig. 1) representations are characterised by the two orbit labels K and k and an UIR of the T^2 subgroup of $U(2)$ which leaves a chosen point on the orbit invariant (centraliser representation). In [6] elements of such representations were called purely magnetic if the centraliser representation is trivial and dyonic otherwise. Moreover we wrote down bases for these representations which have precisely the magnetic (2.10), electric (3.1) and dyonic (3.6) form described here, and which can be realised as wavefunctions on monopole moduli spaces. The representation theory of $U(2) \ltimes \mathbf{R}^4$ does not naturally select orbits with the quantised "height" and "radius" found in the monopole spectrum shown in Fig. 1, and in [6] this requirement had to be imposed by hand. Similarly the condition that the non-abelian magnetic charge directions $\hat{\mathbf{k}}$ of two monopoles or dyons to be multiplied have to be equal or opposite is not natural in the context of $U(2) \ltimes \mathbf{R}^4$ representations, and had to be imposed additionally. However, with these restrictions the Clebsch-Gordan series of $U(2) \ltimes \mathbf{R}^4$ resolves the difficult question of how to combine states from different sectors. In particular it leads to a formula for the tensor product of dyonic states with equal and opposite non-abelian magnetic charges as a superposition of dyonic states $|K; N; j, m\rangle$ in non-abelian electric $\tilde{U}(1) \times U(2)$ representations (K, N, j):

$$
\begin{aligned}
&|K_1, k; N_1, n_1; \hat{\mathbf{k}}\rangle \otimes |K_2, k; N_2, n_2; -\hat{\mathbf{k}}\rangle \\
&\qquad = \sum_{j=|n_1-n_2|}^{\infty} \sum_{m=-j}^{j} \sqrt{2j+1}\, D^j_{m(n_1-n_2)}(P)\ |K_1 + K_2; N_1 + N_2; j, m\rangle,
\end{aligned} \tag{3.10}
$$

where P is parametrised as in (2.11) and (α, β) are again the angles determining the direction of $\hat{\mathbf{k}}$ as in (2.9).

Here we have learnt to think of the magnetic states as coherent states of the magnetic group $\tilde{U}(2)$, which immediately accounts for the quantisation of the magnetic charges K and k. Furthermore the coherency requirement explains the condition of parallel or anti-parallel charges in a tensor product. However, while the introduction of the magnetic group and finally the discussion of the mutually centralising subgroups (3.8) of $\tilde{U}(2) \times U(2)$ leads to a completely satisfactory classification of the various charge sectors found in our theory, the semi-direct product $U(2) \ltimes \mathbf{R}^4$ is indispensable in discussing fusion rules of different sectors. The fusion rules derived from $U(2)\ltimes\mathbf{R}^4$ are only valid in the weak electric coupling regime (which was the context of the discussion in [6]). Using electric-magnetic duality we deduce that the fusion rules in the weak magnetic coupling regime are dictated by the dual semi-direct product $\tilde{U}(2)\ltimes\mathbf{R}^4$, with elements of $(\mathbf{R}^*)^4$ now interpreted as electric charges. Dualising our description of the UIR's of $U(2)\ltimes\mathbf{R}^4$ the reader should have no difficulty in checking that the UIR's of $\tilde{U}(2)\ltimes\mathbf{R}^4$ contain the purely magnetic, coherent electric and dyonic states expected in the weak magnetic coupling regime. Interpreting the states in the weak magnetic coupling regime as representations of $\tilde{U}(2)\ltimes\mathbf{R}^4$ has immediate implications for the fusion rules governing these states. States may again be multiplied under the condition that the directions $\hat{\mathbf{k}}$ characterising the non-abelian electric charge are equal or opposite, and then the outcome is determined by the Clebsch-Gordan coefficients of $\tilde{U}(2)\ltimes\mathbf{R}^4$. The important point, anticipated in the opening paragraph of this section, is that the resulting fusion rules are dual but not equal to the fusion rules in the weak electric coupling regime.

In summary, the fusion rules within each type of sector - non-abelian electric, dyonic and magnetic - are governed by the Clebsch-Gordan series of the groups $\tilde{U}(1)\times U(2)$, $\tilde{T}^2(\hat{\mathbf{k}})\times T^2(\hat{\mathbf{k}})$ and $\tilde{U}(2)\times U(1)$, but to understand fusion between different types of sectors one needs to resort to the semi-direct products $U(2)\ltimes\mathbf{R}^4$ and $\tilde{U}(2)\ltimes\mathbf{R}^4$ at weak electric and weak magnetic coupling respectively. However, even these groups do not tell one how to combine dyons whose associated charge directions $\hat{\mathbf{k}}_1$ and $\hat{\mathbf{k}}_2$ are neither parallel nor anti-parallel. Such a combination does not seem to lead to a well-defined state.

4. Fusion rules for coherent states and non-abelian S-duality

The structure and fusion properties of charge sectors in Yang-Mills-Higgs theory with non-abelian unbroken gauge group is clearly much more intricate than in the abelian example outlined in the introduction. While the picture presented in the previous section enjoys non-abelian electric-magnetic duality we have not yet said anything about the extension of electric-magnetic duality to S-duality. In this section we will show how to do this for coherent states. These states are present in all coupling regimes. Moreover it is straightforward to understand the Witten effect on coherent states. Together with our formulation of non-abelian electric-magnetic duality this leads to an implementation of S-duality on coherent states. As in the abelian example discussed in the introduction, there is a remarkable link between S-duality and fusion rules: the S-duality transformations are automorphisms of the fusion ring of coherent states. We therefore begin our discussion with the fusion properties of coherent states.

Since coherent states are simply special states in the sectors classified in the previous section, their fusion rules can be derived from the result presented there. Rather surprisingly, the truncation of the representation ring of $U(2) \times \mathbf{R}^4$ to electric coherent states is the same as the truncation of the representation ring of $\tilde{U}(2) \times \mathbf{R}^4$ to magnetic coherent states. It is in that sense that coherent states obey universal fusion rules. Here we show that this truncated ring is the representation ring of a certain group associated to the unoriented magnetic/electric charge direction $\pm\hat{\mathbf{k}} \in \mathbf{RP}_2$. For simplicity we will assume in this section that $\hat{\mathbf{k}}$ is the unit vector $(0,0,1)$, so that the tori $T^2(\hat{\mathbf{k}})$ and $\tilde{T}^2(\hat{\mathbf{k}})$ are the standard maximal tori of $U(2)$ and $\tilde{U}(2)$, consisting of diagonal matrices; we denote these standard tori simply by T^2 and $\tilde{T}^2$. The tori associated to general directions $\hat{\mathbf{k}}$ can be obtained from the standard tori by conjugation with an appropriate $SU(2)$ matrix.

Consider for example purely electric coherent states and recall that in the electric strong coupling regime only states with equal or opposite charge directions may be multiplied. In a given UIR (N,j) of $U(2)$ the coherent states with charge direction $\pm\hat{\mathbf{k}} = \pm(0,0,1)$ are precisely the states with highest and lowest $SU(2)$ weight. They can be obtained from each other by acting with (the representative of) the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in U(2)$. Moreover they span an UIR of a subgroup of $U(2)$, namely of the the semi-direct product group

$$S_2 \ltimes T^2, \tag{4.1}$$

where S_2 is the permutation group of two objects, realised canonically as a subgroup of $U(2)$ (with the non-trivial element given above). Clearly we can analogously define a magnetic group $S_2 \ltimes T^2$ whose UIR's contain magnetic coherent states which may be multiplied. More interestingly we can combine these two groups to define the subgroup

$$Coh(2) = S_2 \ltimes (\tilde{T}^2 \times T^2) \tag{4.2}$$

of $\tilde{U}(2) \times U(2)$. Here the permutation group S_2 is realised canonically in the diagonal $U(2)$ subgroup of $\tilde{U}(2) \times U(2)$. The UIR's of (4.2) contain dyonic as well as electric and magnetic coherent states associated with the direction $(0,0,1)$. There are one-dimensional and two-dimensional UIR's. On the one-dimensional UIR's the S_2 action is trivial and at most the abelian central subgroup $\tilde{U}(1) \times U(1)$ acts non-trivially. These representations are thus labelled by integers K and N characterising UIR's of that central subgroup. The two-dimensional UIR's are more interesting for us. They carry additional half-integer labels k and n (characterising the transformation behaviour under the tori $\tilde{T}^1$ and T^1) such that $K+2k$ and $N+2n$ are even. Moreover only the relative sign of k and n matters in the labelling of an UIR, so that either k or n can without loss of generality assumed to be positive. If we take k to be positive we may interpret it as the remnant of the non-abelian magnetic charge (and thus equal to radius of the magnetic orbits found at weak electric coupling). If further n happens to be zero we are in the magnetic situation described above and have a natural identification of the UIR $(K, k, N, n = 0)$ with the span of the two magnetic coherent states associated with the direction $\hat{\mathbf{k}}$. If $n \neq 0$ there is a natural identification of the UIR (K, k, N, n) with the span of of the two dyonic states $|K, k; N, \pm n; \pm\hat{\mathbf{k}}\rangle$ discussed in the previous section (3.6). If, on the other hand, we take n to be positive we may think of it as the remnant of the the non-abelian electric charge (denoted j in Sect. 4). Then, if k happens to be zero we are in the electric situation described above and identify the UIR $(K, k = 0, N, n)$ with the span of the electric coherent states associated with the direction $\hat{\mathbf{k}}$. If $k \neq 0$ we have a natural identification of the UIR (K, k, N, n) with the span of the two dyonic states $|K, \pm k; N, n; \pm\hat{\mathbf{k}}\rangle$.

Having identified the group $Coh(2)$ as the algebraic object whose representations classify coherent and dyonic states associated with a particular direction we show that the fusion properties of these states are encoded in the representation ring of $Coh(2)$. For that purpose, and for a better understanding of the structure of $Coh(2)$, a slightly more general perspective is useful. We therefore briefly consider the situation where the unbroken gauge group is $U(r)$. Then we define $Coh(r)$ analogously as a subgroup of $\tilde{U}(r) \times U(r)$:

$$Coh(r) = S_r \ltimes (\tilde{T}^r \times T^r), \tag{4.3}$$

where $\tilde{T}^r$ and T^r are the canonical maximal tori of $\tilde{U}(r)$ and $U(r)$, and the permutation group S_r (the Weyl group of $SU(r)$) is realised canonically as a subgroup of the diagonal $U(r)$ subgroup of $\tilde{U}(r) \times U(r)$. The most natural tool for discussing the Clebsch-Gordan series of this group are characters, which are central functions on the group. By definition this means that they only depend on the conjugacy class of an element and are therefore effectively functions on

$$M_r = Coh(r)/\text{conjugation} = \text{ Sym}^r \left(U(1) \times U(1)\right). \tag{4.4}$$

Explicitly one can thus think of M_r as an unordered set of pairs of angular coordinates $(\lambda_l, \omega_l) \in [0, 2\pi)^2$ $l = 1, .., r$. As an aside we note that this space also happens to be the moduli space of flat $U(r)$ connections on a torus. This suggests not only how one should generalise the present discussion to other gauge groups but also establishes interesting links with algebraic geometry.

Returning to our example $r = 2$, it is convenient to change to coordinates which explicitly refer to the diagonal rotations $\tilde{U}(1)$ and $U(1)$ and the Cartan subgroups $\tilde{T}^1$ and T^1; these are like "centre of mass" and "relative" coordinates:

$$\begin{aligned} \Lambda &= \frac{\lambda_1 + \lambda_2}{2}, \qquad \lambda = \lambda_1 - \lambda_2 \\ \Omega &= \frac{\omega_1 + \omega_2}{2}, \qquad \omega = \omega_1 - \omega_2. \end{aligned} \tag{4.5}$$

Then we can coordinatise M_2 explicitly as

$$M_2 = \{(\Lambda, \Omega, \lambda, \omega) \in [0, 2\pi)^2 \times [-2\pi, 2\pi)^2\} / \sim \tag{4.6}$$

where the equivalence relation $\sim$ identifies $(\Lambda, \Omega, \lambda, \omega)$ with $(\Lambda, \Omega, -\lambda, -\omega)$ and $(\Lambda, \Omega, \lambda, \omega)$ with $(\Lambda + \pi, \Omega + \pi, \lambda + 2\pi, \omega + 2\pi)$.

The character of the $Coh(2)$-representation (K, k, N, n) is then the function

$$\chi_{K,k,N,n}(\Lambda, \Omega, \lambda, \omega) = e^{i(K\Lambda + N\Omega)} \cos(k\lambda + n\omega). \tag{4.7}$$

It follows from the general theory of characters (and can easily be checked explicitly) that the set of $Coh(2)$ characters form an orthonormal basis of $L^2(M_2)$. Then the Clebsch-Gordan series of $Coh(2)$ can be read of from pointwise multiplication of the characters and subsequent expansion in the basis $\chi_{K,k,N,n}$. This yields the following fusion rules for the sectors (K, k, N, n):

$$\begin{aligned} (K_1, k_1, N_1, n_1) \otimes (K_2, k_2, N_2, n_2) = {} & (K_1 + K_2, k_1 + k_1, N_1 + N_2, n_1 + n_2) \\ & \oplus (K_1 + K_2, k_1 - k_1, N_1 + N_2, n_1 - n_2). \end{aligned} \tag{4.8}$$

These are the promised "universal" fusion rules for coherent and dyonic states associated with a given unoriented vector $\pm\hat{\mathbf{k}}$. In particular they agree with the fusion rules one obtains from tensoring coherent or dyonic states according to the representation theory of $U(2) \ltimes \mathbf{R}^4$ or $\tilde{U}(2) \ltimes \mathbf{R}^4$ (with the (anti-)parallelity condition on the charge direction $\hat{\mathbf{k}}$) and subsequently truncating to coherent and dyonic states.

The challenge of formulating non-abelian S-duality consists of combining our formulation of electric-magnetic duality with an implementation of the Witten effect such that the two generate an $SL(2,\mathbf{Z})$ action. In this wording of the task it is a priori not even clear whether we should aim for an action on charge sectors or on individual quantum states (in the abelian situation where all UIR's are one-dimensional these two possibilities coincide). Here we are going to propose an $SL(2,\mathbf{Z})$-action which maps the coherent states in one charge sector onto the coherent states of another. The central role of coherent states stems from the fact that they are present in all coupling regimes. A practical advantage is that we have no difficulty implementing the Witten effect on coherent states. For suppose we have a purely magnetic coherent state $|K;k,\hat{\mathbf{k}}\rangle$. In the weak electric coupling regime we identify it with a classical monopole with charge $(K,\mathbf{k})$. A simple extension of the original calculation performed by Witten [7] shows that a shift in the θ-angle by 2π transforms this state into a dyon as follows:

$$|K;k,\hat{\mathbf{k}}\rangle \to |K,k;K,k;\hat{\mathbf{k}}\rangle. \tag{4.9}$$

The key to the implementation of full S-duality on coherent states is a natural $SL(2,\mathbf{Z})$ action on M_2. In order to indicate the generality of the construction we write down this action for M_r. An element of $SL(2,\mathbf{Z})$, written as in (1.2), acts on each pair (λ_l,ω_l) in the parametrisation (4.4) of M_r according to

$$(\lambda_l,\omega_l) \to (d\lambda_l - b\omega_l, -c\lambda_l + a\omega_l). \tag{4.10}$$

Since the action is the same for each pair it clearly commutes with the action of the permutation group S_r on the set of pairs $\{(\lambda_\rho,\omega_\rho)\}_{\rho=1,\ldots,r}$ and is thus well-defined on M_r. The action of the modular group on the manifold M_r induces an action on functions on M_r and in particular therefore on characters of the group $Coh(r)$. We explicitly describe this again in the case $r=2$.

In that case we find in particular the following action of the generator S of electric-magnetic duality

$$(K,k,N,n) \to (-N,-n,K,k), \tag{4.11}$$

which should be combined with the inversion of the complex coupling constant $\tau \to -1/\tau$. Recall that only the relative sign of k and n matters in the labelling of $Coh(2)$ representations; the action of S changes this relative sign. The generator T implements the Witten effect in the required fashion

$$(K, k, N, n) \to (K, k, N + K, n + k). \tag{4.12}$$

While these formulae are very similar to the formulae given for abelian S-duality in Sect. 1 we emphasise that they refer to (in general) two-dimensional UIR's of $Coh(2)$. Applied to non-abelian magnetic states for example (4.11) maps a doublet of magnetic coherent states onto a doublet of electric coherent states and (4.12) maps a doublet of magnetic coherent states onto a doublet of dyonic states. In the more general case of $U(r)$ as unbroken gauge group our formalism would yield a map between higher-dimensional (at most r-dimensional) sets of magnetic, electric and dyonic states. These sets are in one-to-one correspondence with orbits of the Weyl group S_r of $SU(r)$. Weyl orbits have played a role in earlier discussions of duality [2] and here we find them back as sets of coherent and/or dyonic states which may be multiplied consistently at all values of the coupling e.

Finally we note that, as in the abelian case, S-duality transformations are automorphisms of the fusion rules. This follows automatically from our encoding of the fusion rules in the pointwise multiplication of characters which commutes with the $SL(2, \mathbf{Z})$-action on the arguments of the characters.

5. Conclusion

The realisation of symmetry and the implementation of S-duality in Yang-Mills theory with non-abelian residual symmetry is richer and more intricate than in the abelian situation. In this concluding section we highlight three important general points of our discussion.

The first point concerns the charge sectors. While both electric and magnetic non-abelian symmetry can be found in the theory, they are never fully realised at the same time. The allowed charge sectors are classified by UIR's of pairs of commuting subgroups of the electric and magnetic symmetry groups, as displayed explicitly in (3.8).

The second point concerns the fusion rules. Within a given type of sector these are given by the representation ring of a group. However fusing different types of sectors is complicated and depends on the coupling regime. In our case, fusion rules were encoded in the representation ring of the semi-direct product group $U(2) \ltimes \mathbf{R}^4$ at weak electric coupling but in the representation ring of the dual $\tilde{U}(2) \ltimes \mathbf{R}^4$ at weak magnetic coupling.

The third point concerns duality. Our classification of sectors and their fusion rules enjoy manifest non-abelian electric-magnetic duality, but to implement S-duality we restricted attention to coherent states. These are special in that they appear in all coupling regimes and have universal fusion properties in the sense that their fusion ring is a sub-ring of the representation ring of both $U(2) \ltimes \mathbf{R}^4$ and $\tilde{U}(2) \ltimes \mathbf{R}^4$. On coherent states S-duality can be implemented as an $SL(2, \mathbf{Z})$ action on charge sectors which leaves the fusion rules invariant. Conversely one could define S-duality as the automorphism group of the fusion ring of coherent states.

There are a number of further questions which arise from our discussion. At the technical level one would like to generalise to other gauge groups, particularly ones which are not simply laced. At a deeper conceptual level one would like to find a reason for the "freezing" of magnetic (electric) degrees of freedom into coherent states at electric (magnetic) weak coupling. One would also like to say more about the intermediate range $e \approx 1$, where fusion properties are captured by neither the electric $U(2) \ltimes \mathbf{R}^4$ nor the magnetic $\tilde{U}(2) \ltimes \mathbf{R}^4$. While we are confident to be able to report on the technical issue of general gauge groups in the near future, the conceptual questions pose a deeper challenge.

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