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Mapping class group actions in Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$

C. Meusburger[1] and B. J. Schroers[2]
Department of Mathematics, Heriot-Watt University
Edinburgh EH14 4AS, United Kingdom

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Abstract

We study the action of the mapping class group of an oriented genus g surface with n punctures and a disc removed on a Poisson algebra which arises in the combinatorial description of Chern-Simons gauge theory when the gauge group is a semidirect product $G \ltimes \mathfrak{g}^*$. We prove that the mapping class group acts on this algebra via Poisson isomorphisms and express the action of Dehn twists in terms of an infinitesimally generated G-action. We construct a mapping class group representation on the representation spaces of the associated quantum algebra and show that Dehn twists can be implemented via the ribbon element of the quantum double $D(G)$ and the exchange of punctures via its universal R-matrix.

1 Introduction

This paper investigates a mapping class group action on a certain Poisson algebra and on the representation spaces of the associated quantum algebra. This Poisson algebra, in the following referred to as flower algebra, plays an important role in the combinatorial description of the phase space of Chern-Simons theory with semidirect product gauge groups of the form $G \ltimes \mathfrak{g}^*$, where G is a Lie group, $\mathfrak{g}^*$ the dual of its Lie algebra and G acts on $\mathfrak{g}^*$ in the coadjoint representation. Its classical structure and quantisation were studied in an earlier paper [23].

Although mapping class groups and Chern-Simons gauge theories are research topics in their own right, our interest in them is motivated by their relevance to physics. Chern-Simons gauge theory with gauge group $G \ltimes \mathfrak{g}^*$ occurs in the Chern-Simons formulation of (2+1)-dimensional gravity with vanishing cosmological constant [25], where, depending

[1] cm@ma.hw.ac.uk
[2] bernd@ma.hw.ac.uk

on the signature of the spacetime, the gauge group is the three dimensional Poincaré or Euclidean group. For spacetimes of topology $\mathbb{R} \times S$, where S is an oriented surface of arbitrary genus, possibly with punctures and boundary components, large diffeomorphisms of the surface S give rise to Poisson symmetries or canonical transformations on the phase space [22]. There is evidence suggesting that these symmetries play an important role in the physical interpretation of the theory, in particular for the dynamics of gravitationally interacting particles [12, 8].

The flower algebra emerges in a description of the moduli space of flat connections on a genus g surface $S_{g,n}$ with n punctures, discovered by Fock and Rosly [14] and developed further by Alekseev, Grosse and Schomerus [3, 4, 5]. In this description, the moduli space is given as a quotient of the space of holonomies associated to a set of $n+2g$ generators of the surface's fundamental group equipped with a certain Poisson structure. In our case, these holonomies are elements of $G \ltimes \mathfrak{g}^*$, and one obtains a Poisson structure on the manifold $(G \ltimes \mathfrak{g}^*)^{n+2g}$. The flower algebra is the algebra of a particular class of functions on $(G \ltimes \mathfrak{g}^*)^{n+2g}$ with this Poisson bracket. In [23], we investigated the classical properties of this Poisson algebra and constructed the corresponding quantum algebra and its irreducible Hilbert space representations. In this paper, we show that the mapping class group $\mathrm{Map}(S_{g,n} \backslash D)$ of the surface $S_{g,n}$ with a disc D removed acts on the flower algebra and determine the associated quantum action.

We prove that the mapping class group action on the flower algebra is a Poisson action and show that the action of Dehn twists around embedded curves on the surface $S_{g,n} \backslash D$ can be expressed in terms of a G-action that is infinitesimally generated via the Poisson bracket. For the case where the exponential map $\exp : \mathfrak{g} \to G$ is surjective, we give explicit Hamiltonians whose flow by one unit is equal to the action of Dehn twists. These flows are special examples of the Hamiltonian "twist flows" studied by Goldman in [16], but the particular Hamiltonians we consider and their relation with Dehn twists appear to be new.

We then demonstrate how these classical features are mirrored by corresponding structures in the quantum theory. We show that elements of the mapping class group act as algebra automorphisms on the quantised flower algebra and implement this action on its representation spaces. This allows us to relate the quantum action of the mapping class group to different representations of the quantum double $D(G)$ of the group G. We find a canonical way of associating a representation of the quantum double $D(G)$ to each embedded curve, such that the action of the corresponding Dehn twist is given by the ribbon element of $D(G)$. We find an implementation of the exchange of punctures on the surface $S_{g,n} \backslash D$ in terms of the action of the universal R-matrix in the tensor product of two representations of $D(G)$, familiar from the theory of quantum groups.

The paper is structured as follows: Sect. 2 gives a summary of our results in [23] required for the understanding of this article, which is necessarily rather condensed. In Sect. 3 we discuss the mapping class group action on the classical flower algebra and express the action of Dehn twists in terms of infinitesimally generated group actions as outlined above. Sect. 4 investigates the corresponding quantum action and relates it to representations of the quantum double $D(G)$, followed by a brief outlook in Sect. 5. The appendix lists a set of generators of the mapping class group and their actions on the fundamental group $\pi_1(S_{g,n}\backslash D)$.

2 The phase space of Chern-Simons gauge theory with gauge group $G \ltimes \mathfrak{g}^*$ and the flower algebra

2.1 Notation and conventions

We consider groups $G \ltimes \mathfrak{g}^*$ which are the semidirect product of a finite-dimensional, simply connected and connected Lie group G and the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$, viewed as an abelian group. All Lie algebras are vector spaces over $\mathbb{R}$ unless stated otherwise, and Einstein summation convention is used throughout the paper. Following the conventions of [20], we define $\mathrm{Ad}^*(g)$ to be the algebraic dual of $\mathrm{Ad}(g)$, i. e.

$$\langle \mathrm{Ad}^*(g)\boldsymbol{j}, \xi\rangle = \langle \boldsymbol{j}, \mathrm{Ad}(g)\xi\rangle \qquad \forall \boldsymbol{j} \in \mathfrak{g}^*, \xi \in \mathfrak{g}, g \in G, \tag{2.1}$$

so that the coadjoint action of $g \in G$ is given by $\mathrm{Ad}^*(g^{-1})$. Writing elements of $G \ltimes \mathfrak{g}^*$ as $(u, \boldsymbol{a})$ with $u \in G$ and $\boldsymbol{a} \in \mathfrak{g}^*$, we have the group multiplication law

$$(u_1, \boldsymbol{a}_1) \cdot (u_2, \boldsymbol{a}_2) = (u_1 \cdot u_2, \boldsymbol{a}_1 + \mathrm{Ad}^*(u_1^{-1})\boldsymbol{a}_2). \tag{2.2}$$

We also use the parametrisation

$$(u, \boldsymbol{a}) = (u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) \qquad \text{with } u \in G,\ \boldsymbol{a}, \boldsymbol{j} \in \mathfrak{g}^*, \tag{2.3}$$

where, as explained in [23], the pair $(u, \boldsymbol{j})$ should be thought of as an element of the dual Poisson-Lie group.

Let J_a, P^a, $a = 1, \ldots, \dim G$, denote the generators of the Lie algebra $\mathrm{Lie}\,(G \ltimes \mathfrak{g}^*) = \mathfrak{g} \oplus \mathfrak{g}^*$, such that the generators J_a form a basis of $\mathfrak{g} = \mathrm{Lie}\, G$ and the generators P^a a basis of $\mathfrak{g}^*$. Then the commutator of the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$ is given by

$$[J_a, J_b] = f_{ab}^{\ \ c}\, J_c \qquad [J_a, P^b] = -f_{ac}^{\ \ b}\, P^c \qquad [P^a, P^b] = 0, \tag{2.4}$$

where $f_{ab}^{\ \ c}$ are the structure constants of $\mathfrak{g}$. We denote by J_a^R, J_a^L the left-and right-invariant vector fields on G associated to the generators J_a

$$J_a^R F(u) := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} F(ue^{tJ_a}) \qquad J_a^L F(u) := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} F(e^{-tJ_a}u) \quad \forall u \in G,\ F \in \mathcal{C}^\infty(G). \tag{2.5}$$

2.2 The flower algebra for gauge group $G \ltimes \mathfrak{g}^*$

The flower algebra plays an important role in the description of the phase space of Chern-Simons theory with semidirect product gauge groups of type $G \ltimes \mathfrak{g}^*$. Mathematically, this phase space is the moduli space of flat $G \ltimes \mathfrak{g}^*$-connections on the surface $S_{g,n}$, and it can be described as a quotient of the space of holonomies holonomies associated to a set of generators of the surface's fundamental group. Fock and Rosly [14] and Alekseev, Grosse and Schomerus [3, 4, 5] defined a Poisson structure on the space of holonomies, which, via Poisson reduction, gives rise to the canonical Poisson structure on the moduli space [15, 6]. For Chern-Simons theory with compact, semisimple gauge groups, this Poisson structure on the space of holonomies was investigated by Alekseev, Grosse and Schomerus [3, 4, 5] and quantised via their formalism of combinatorial quantisation of Chern-Simons gauge theories. The case of (non-compact and non-semisimple) semidirect product gauge groups of type $G \ltimes \mathfrak{g}^*$ was studied in [23], where we discussed the properties of this Poisson structure and developed a quantisation procedure.

In order to define the flower algebra for Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$ on a punctured surface, we need a set of generators of the surface's fundamental group. Both the fundamental group $\pi_1(S_{g,n})$ of a genus g surface $S_{g,n}$ with n punctures and the fundamental group of the associated surface $S_{g,n}\setminus D$ with a disc D removed is generated by the equivalence classes loops m_i, $i = 1, \ldots, n$, around the punctures and two curves a_j, b_j, $j = 1, \ldots, g$, for each handle, shown in Fig. 1. In the case of the surface $S_{g,n}\setminus D$ with a disc removed they generate the fundamental group freely, whereas for the surface $S_{g,n}$ they are subject to the relation

$$[b_g, a_g^{-1}] \cdot \ldots \cdot [b_1, a_1^{-1}] \cdot m_n \cdot \ldots \cdot m_1 = 1, \quad \text{with} \quad [b_i, a_i^{-1}] = b_i a_i^{-1} b_i^{-1} a_i. \tag{2.6}$$

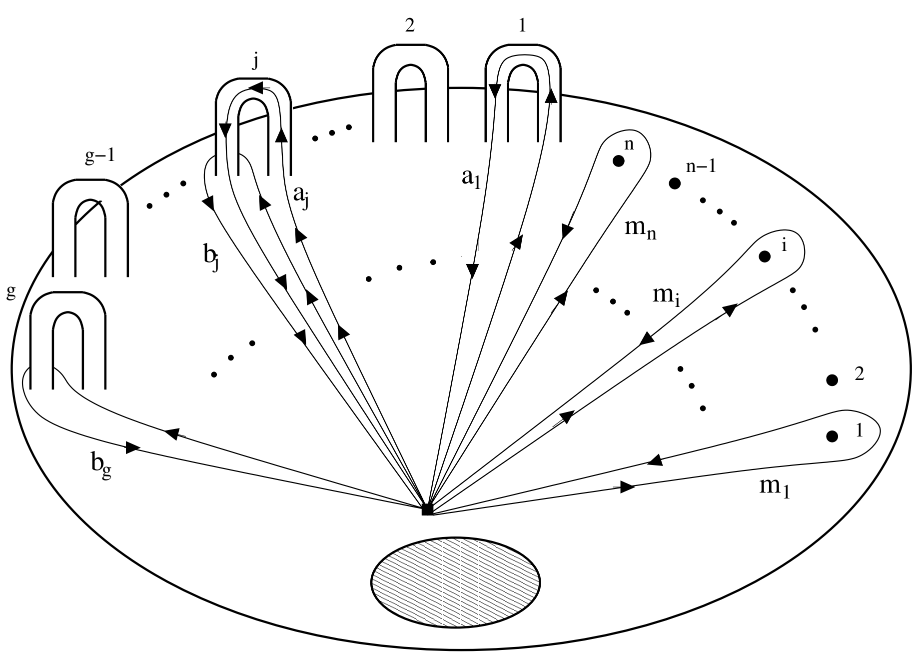

Fig. 1

The generators of the fundamental group of the surface $S_{g,n}\setminus D$

In the rest of the paper, we do not distinguish notationally between closed curves on $S_{g,n}$ and $S_{g,n}\backslash D$, and their equivalence classes in the fundamental group $\pi_1(S_{g,n})$ and $\pi_1(S_{g,n}\backslash D)$.

Whereas the holonomies $A_j = \mathrm{Hol}(a_j)$, $B_j = \mathrm{Hol}(b_j)$ associated to each handle are general elements of the group $G \ltimes \mathfrak{g}^*$, the holonomies $M_i = \mathrm{Hol}(m_i)$ around the punctures lie in fixed $G \ltimes \mathfrak{g}^*$-conjugacy classes

$$\mathcal{C}_{\mu_i s_i} = \{(v, \boldsymbol{x}) \cdot (g_{\mu_i}, -\boldsymbol{s}_i) \cdot (v, \boldsymbol{x})^{-1} \mid (v, \boldsymbol{x}) \in G \ltimes \mathfrak{g}^*\}. \tag{2.7}$$

For a geometrical interpretation of the labels μ_i and s_i we refer the reader to [23].

By applying the work of Fock and Rosly [14] and Alekseev, Grosse and Schomerus [3, 4, 5] to the case of gauge group $G \ltimes \mathfrak{g}^*$, one obtains a Poisson structure on $(G \ltimes \mathfrak{g}^*)^{n+2g}$. However, for gauge groups $G \ltimes \mathfrak{g}^*$ it is more convenient to work with the slightly different formulation used in [23]. We parametrise the holonomies according to (2.3) as $X = \mathrm{Hol}(x) = (u_X, -\mathrm{Ad}^*(u_X^{-1})\boldsymbol{j}^X)$ for $X \in \{M_1, \ldots, M_n, A_1, B_1, \ldots, A_g, B_g\}$, expand the vectors $\boldsymbol{j}^X$ as $\boldsymbol{j}^X = j_b^X P^b$ and denote by the same symbol the coordinate functions

$$j_a^X \in \mathcal{C}^\infty((G \ltimes \mathfrak{g}^*)^{n+2g}) : (M_1, \ldots, M_n, A_1, B_1, \ldots, A_g, B_g) \mapsto j_a^X. \tag{2.8}$$

Instead of the algebra $\mathcal{C}^\infty((G \ltimes \mathfrak{g}^*)^{n+2g})$ we then consider the algebra generated by the functions in $\mathcal{C}^\infty(G^{n+2g})$ together with these maps j_a^X with the Poisson structure given below.

Definition 2.1 *(Flower algebra for groups $G \ltimes \mathfrak{g}^*$)*

The flower algebra $\mathcal{F}$ for gauge group $G \ltimes \mathfrak{g}^*$ on a genus g surface $S_{g,n}$ with n punctures is the commutative Poisson algebra

$$\mathcal{F} = S\left(\bigoplus_{k=1}^{n+2g} \mathfrak{g}\right) \otimes \mathcal{C}^\infty(G^{n+2g}), \tag{2.9}$$

where $S\left(\bigoplus_{k=1}^{n+2g} \mathfrak{g}\right)$ is the symmetric envelope of the real Lie algebra $\bigoplus_{k=1}^{n+2g} \mathfrak{g}$, i.e. the polynomials with real coefficients on the vector space $\bigoplus_{l=1}^{n+2g} \mathfrak{g}^*$. In terms of a fixed basis $\mathcal{B} = \{j_a^{M_i}, j_a^{A_k}, j_a^{B_k},\ i=1,\ldots,n,\ \ k=1,\ldots,g,\ \ a=1,\ldots,\dim G\}$, its Poisson structure is given by

$$\begin{aligned}
&\{j_a^X \otimes 1, j_b^X \otimes 1\} = -f_{ab}{}^c\, j_c^X \otimes 1 \\
&\{j_a^X \otimes 1, j_b^Y \otimes 1\} = -f_{db}{}^c\, j_c^Y \otimes (\delta_a{}^d - Ad^*(u_X)_a{}^d) \qquad && \forall X, Y \in \{M_1, \ldots, B_g\},\ X < Y \\
&\{j_a^{A_i} \otimes 1, j_b^{B_i} \otimes 1\} = -f_{ab}{}^c\, j_c^{B_i} \otimes 1 && \forall i = 1, \ldots, g
\end{aligned}$$

$$\{j_a^{M_i} \otimes 1, 1 \otimes F\} = -1 \otimes (J_a^{R_{M_i}} + J_a^{L_{M_i}})F - 1 \otimes (\delta_a{}^b - Ad^*(u_{M_i})_a{}^b)\left(\sum_{Y > M_i} (J_b^{R_Y} + J_b^{L_Y})F\right)$$

$$
\begin{aligned}
\{j_a^{A_i} \otimes 1, 1 \otimes F\} = & -1 \otimes (J_a^{R_{A_i}} + J_a^{L_{A_i}})F - 1 \otimes (J_a^{R_{B_i}} + J_a^{L_{B_i}})F - 1 \otimes Ad^*(u_{B_i}^{-1} u_{A_i})_a{}^b J_b^{R_{B_i}} \\
& - 1 \otimes (\delta_a{}^b - Ad^*(u_{A_i})_a{}^b) \left(\sum_{Y > A_i} (J_b^{R_Y} + J_b^{L_Y}) F \right) \\
\{j_a^{B_i} \otimes 1, 1 \otimes F\} = & -1 \otimes J_a^{L_{A_i}} F - 1 \otimes (J_a^{R_{B_i}} + J_a^{L_{B_i}})F \\
& - 1 \otimes (\delta_a{}^b - Ad^*(u_{B_i})_a{}^b) \left(\sum_{Y > B_i} (J_b^{R_Y} + J_b^{L_Y}) F \right), \qquad (2.10)
\end{aligned}
$$

where $F \in \mathcal{C}^\infty(G^{n+2g})$, $M_1 < \ldots < M_n < A_1, B_1 < \ldots < A_g, B_g$ and $J_a^{L_X}$, $J_a^{R_X}$ denote the right- and left invariant vector fields (2.5) *on the different copies of G.*

Note that this definition does not restrict the holonomies M_i associated to the punctures to fixed $G \ltimes \mathfrak{g}^*$-conjugacy classes $\mathcal{C}_{\mu_i s_i}$. Instead, these conjugacy classes arise as the symplectic leaves of the Poisson structure (2.10) on $(G \ltimes \mathfrak{g}^*)^{n+2g}$. Furthermore, we showed in [23] by extending the work of Alekseev and Malkin [2] to groups of type $G \ltimes \mathfrak{g}^*$ that the Poisson structure on the symplectic leaves is given by a symplectic potential Θ on $(G \ltimes \mathfrak{g}^*)^{n+2g}$. This potential can be expressed in terms of the holonomies as follows

Theorem 2.2 *(Symplectic leaves and decoupling)*

The symplectic leaves of the Poisson manifold $(G \ltimes \mathfrak{g}^*)^{n+2g}$ with bracket (2.10) *are of the form $\mathcal{C}_{\mu_1 s_1} \times \ldots \times \mathcal{C}_{\mu_n s_n} \times T^*(G)^{2g}$, where $\mathcal{C}_{\mu_i s_i}$ denote $G \ltimes \mathfrak{g}^*$-conjugacy classes as in* (2.7). *Let $\omega_{\mathcal{F}}$ denote the symplectic form on these symplectic leaves, define a map $\beta : G^{n+2g} \to G^{n+2g}$*

$$
\begin{aligned}
\beta : (v_{M_1}, \ldots, v_{M_n}, u_{A_1}, \ldots, u_{B_g}) \mapsto & (v_{M_1} g_{\mu_1} v_{M_1}^{-1}, \ldots, v_{M_n} g_{\mu_n} v_{M_n}^{-1}, u_{A_1}, \ldots, u_{B_g}) \qquad (2.11) \\
& =: (\beta_{M_1}(v_{M_1}, \ldots, u_{B_g}), \ldots, \beta_{B_g}(v_{M_1}, \ldots, u_{B_g}))
\end{aligned}
$$

and extend it trivially to a map $\tau : (G \ltimes \mathfrak{g}^*)^{n+2g} \to (G \ltimes \mathfrak{g}^*)^{n+2g}$ via

$$
\begin{aligned}
\tau : & (v_{M_1}, \boldsymbol{j}^{M_1}, \ldots, v_{M_n}, \boldsymbol{j}^{M_n}, u_{A_1}, \boldsymbol{j}^{A_1}, \ldots, u_{B_g}, \boldsymbol{j}^{B_g}) \mapsto \qquad (2.12) \\
& (\beta_{M_1}(v_{M_1}, \ldots, u_{B_g}), \boldsymbol{j}^{M_1}, \ldots, \beta_{B_g}(v_{M_1}, \ldots, u_{B_g}), \boldsymbol{j}^{B_g}).
\end{aligned}
$$

Then, the pull-back $\tau^* \omega_{\mathcal{F}}$ of $\omega_{\mathcal{F}}$ with τ coincides with the exterior derivative of the symplectic potential

$$
\begin{aligned}
\Theta =& \sum_{i=1}^{n} \langle\, d(u_{M_{i-1}} \cdots u_{M_1})(u_{M_{i-1}} \cdots u_{M_1})^{-1} - dv_{M_i} v_{M_i}^{-1} \,,\, j_a^{M_i} P^a \rangle \qquad (2.13)\\
&+ \sum_{i=1}^{g} \langle\, d(u_{K_{i-1}} \cdots u_{M_1})(u_{K_{i-1}} \cdots u_{M_1})^{-1} \,,\, j_a^{A_i} P^a \rangle\\
&\qquad - \langle\, d(u_{A_i}^{-1} u_{B_i}^{-1} u_{A_i} u_{K_{i-1}} \cdots u_{M_1})(u_{A_i}^{-1} u_{B_i}^{-1} u_{A_i} u_{K_{i-1}} \cdots u_{M_1})^{-1} \,,\, j_a^{A_i} P^a \rangle\\
&+ \sum_{i=1}^{g} \langle\, d(u_{A_i}^{-1} u_{B_i}^{-1} u_{A_i} u_{K_{i-1}} \cdots u_{M_1})(u_{A_i}^{-1} u_{B_i}^{-1} u_{A_i} u_{K_{i-1}} \cdots u_{M_1})^{-1} \,,\, j_a^{B_i} P^a \rangle\\
&\qquad - \langle\, d(u_{B_i}^{-1} u_{A_i} u_{K_{i-1}} \cdots u_{M_1})(u_{B_i}^{-1} u_{A_i} u_{K_{i-1}} \cdots u_{M_1})^{-1} \,,\, j_a^{B_i} P^a \rangle,
\end{aligned}
$$

where $u_{K_i} = [u_{B_i}, u_{A_i}^{-1}] = u_{B_i} u_{A_i}^{-1} u_{B_i}^{-1} u_{A_i}$.

Proof: This follows from Theorems 2.4., 2.5. in [23] by expressing the symplectic form Θ defined there in terms of the the coordinate functions j_a^X (2.8). □

Under the pull-back τ^* the conjugation action on the group elements associated to the punctures gets mapped to left-multiplication. Hence, if we consider the Poisson algebra generated by the maps j_a^X in (2.8) and functions in $\mathcal{C}^\infty(G^{n+2g})$ with the bracket induced by the symplectic potential Θ (2.13) we obtain a modified bracket $\{\,,\,\}_\Theta$ on $S\left(\bigoplus_{k=1}^{n+2g} \mathfrak{g}\right) \otimes \mathcal{C}^\infty(G^{n+2g})$ that is given by (2.10) with the exception of

$$
\begin{aligned}
\{j_a^{M_i} \otimes 1, 1 \otimes F\}_\Theta =& - 1 \otimes J_a^{L_{M_i}} F - 1 \otimes (\delta_a{}^b - \mathrm{Ad}^*(u_{M_i})_a{}^b) \left(\sum_{j=i+1}^{n} J_b^{L_{M_j}} F \right) \qquad (2.14)\\
&- 1 \otimes (\delta_a{}^b - \mathrm{Ad}^*(u_{M_i})_a{}^b) \left(\sum_{j=1}^{g} (J_b^{R_{A_j}} + J_b^{L_{A_j}} + J_b^{R_{B_j}} + J_b^{L_{B_j}}) F \right).
\end{aligned}
$$

In [23], we made use of this link between the flower algebra Poisson structure (2.10) and the symplectic potential Θ (2.13) to construct the corresponding quantum algebra and to investigate its representation theory. We obtained the following theorem

Theorem 2.3 *(Quantum flower algebra)*

The quantum algebra for the flower algebra in Def. 2.9 is the associative algebra

$$
\hat{\mathcal{F}} = U \left(\bigoplus_{k=1}^{n+2g} \mathfrak{g} \right) \otimes \mathcal{C}^\infty(G^{n+2g}, \mathbb{C}), \qquad (2.15)
$$

with the multiplication defined by

$$
(\xi \otimes F) \cdot (\eta \otimes K) = \xi \cdot_U \eta \otimes FK + i\hbar\, \eta \otimes F\{\xi \otimes 1, 1 \otimes K\}, \qquad (2.16)
$$

where $\xi, \eta \in \bigoplus_{k=1}^{n+2g} \mathfrak{g}$, $F, K \in \mathcal{C}^\infty(G^{n+2g}, \mathbb{C})$ *and* $\cdot_U$ *denotes the multiplication in* $U\left(\bigoplus_{k=1}^{n+2g} \mathfrak{g}\right)$. *The bracket* $\{\,,\,\}$ *is given by* (2.10).

We found in [23] that the representation theory of this algebra is best investigated in the framework of representation theory of transformation group algebras. As the discussion is quite technical, we summarise only the main result and refer the reader to [23] for further details and some technical assumptions on the group G. Further information about transformation group algebras can be found in [18], which gives a treatment closely related to our situation.

Theorem 2.4 *(Representations of the quantum flower algebra)*

Under the technical assumptions on the group G in [23], the irreducible representations of the flower algebra are labelled by n G-conjugacy classes $\mathcal{C}_{\mu_i} = \{g g_{\mu_i} g^{-1} | g \in G\}$, $i = 1, \ldots, n$, and irreducible unitary Hilbert space representations $\Pi_{s_i} : N_{\mu_i} \to V_{s_i}$ of the stabilisers $N_{\mu_i} = \{g \in G | g g_{\mu_i} g^{-1} = g_{\mu_i}\}$ of chosen elements $g_{\mu_1}, \ldots g_{\mu_n}$ in those conjugacy classes. Consider the space

$$\begin{aligned} L^2_{\mu_1 s_1 \ldots \mu_n s_n} &= \big\{\psi : G^{n+2g} \to V_{s_1} \otimes \ldots \otimes V_{s_n}) \mid \psi(v_1 h_1, \ldots, v_{M_n} h_n, u_{A_1}, \ldots, u_{B_g}) \\ &= (\Pi_{s_1}(h_1^{-1}) \otimes \ldots \otimes \Pi_{s_n}(h_n^{-1}))\, \psi(v_{M_1}, \ldots, v_{M_n}, u_{A_1}, \ldots, u_{B_g})\ \forall h_i \in N_{\mu_i}\ and ||\psi||^2 < \infty \big\}, \end{aligned}$$

with inner product

$$\begin{aligned} \langle \psi, \phi \rangle = \int_{G/N_{\mu_1} \times \ldots \times G/N_{\mu_n} \times G^{2g}} &(\,\psi\,,\,\phi\,)\,(v_{M_1}, \ldots, v_{M_n}, u_{A_1}, \ldots u_{B_g}) \\ &dm_1(v_{M_1} \cdot N_1) \cdots dm_n(v_{M_n} \cdot N_n) \cdot du_{A_1} \cdots du_{B_g}, \end{aligned} \tag{2.17}$$

where $(\ ,\ )$ is the canonical inner product on the tensor product of Hilbert spaces $V_{s_1} \otimes \ldots \otimes V_{s_n}$. Then the representation spaces $V_{\mu_1 s_1 \ldots \mu_n s_n}$ are obtained from $L^2_{\mu_1 s_1 \ldots \mu_n s_n}$ by dividing out the zero-norm states. The quantum flower algebra acts on a dense subspace $V^{\infty}_{\mu_1 s_1 \ldots \mu_n s_n}$ of C^∞-vectors [23] according to

$$\begin{aligned} \Pi_{\mu_1 s_1 \ldots \mu_n s_n}(X \otimes F)\psi &= -i\hbar (F \circ \beta) \cdot \{X, \psi\}_\Theta \\ \Pi_{\mu_1 s_1 \ldots \mu_n s_n}(1 \otimes F)\psi &= (F \circ \beta) \cdot \psi \end{aligned} \tag{2.18}$$

where $F \in \mathcal{C}^\infty(G^{n+2g})$, $X \in \mathfrak{g}^{n+2g}$ and β is given by (2.11).

3 The classical action of the mapping class group

3.1 Poisson action of the mapping class group

As a diffeomorphism invariant theory, Chern-Simons theory on $\mathbb{R} \times S_{g,n}$ is in particular invariant under orientation preserving diffeomorphisms of $S_{g,n}$ which are not connected to the identity. The equivalence classes of such diffeomorphisms constitute the surface's mapping class group $\mathrm{Map}(S_{g,n})$ [10]. Therefore, one would expect the mapping class group to act via Poisson isormophisms on the phase space of Chern-Simons theory, the moduli space of flat connections. The moduli space is obtained from the set of holonomies around the curves in Fig. 1 by imposing the relation (2.6) and dividing by conjugation. In the flower algebra, by contrast, the relation (2.6) is not imposed and therefore there is no group action of $\mathrm{Map}(S_{g,n})$ on the flower algebra. We shall now show that, instead, there is a a group action of the mapping class group $\mathrm{Map}(S_{g,n}\backslash D)$ on the flower algebra, and that this action is Poisson.

The mapping class group $\mathrm{Map}(S_{g,n}\backslash D)$ is the group of equivalence classes of orientation preserving diffeomorphisms of $S_{g,n}\backslash D$ which fix the punctures as a set and the boundary of the disc D pointwise; diffeomorphisms are equivalent if they differ by one which is isotopic to the identity. It contains elements that leave the punctures invariant as well as elements that exchange different punctures. The former, by definition, form the pure mapping class group $\mathrm{PMap}(S_{g,n}\backslash D)$ related to the mapping class group by the short exact sequence

$$1 \to \mathrm{PMap}(S_{g,n}\backslash D) \xrightarrow{i} \mathrm{Map}(S_{g,n}\backslash D) \xrightarrow{\pi} S_n \to 1, \tag{3.1}$$

where i is the canonical embedding and $\pi : \mathrm{Map}(S_{g,n}\backslash D) \to S_n$ the projection onto the symmetric group that assigns to each element of the mapping class group the associated permutation of the punctures. As explained in [9, 10], the pure mapping class group $\mathrm{PMap}(S_{g,n}\backslash D)$ is generated by Dehn twists around a set of embedded curves, and a set of generators of the full mapping class group $\mathrm{Map}(S_{g,n})$ can be obtained by supplementing this set with $n-1$ elements which get mapped to the elementary transpositions via π. A set of generators of the pure and full mapping class group and their action on the fundamental group is given in in the appendix.

The action of the mapping class group on the flower algebra arises in the following way. Elements $\lambda \in \mathrm{Map}(S_{g,n}\backslash D)$ act as automorphisms on the fundamental group $\pi_1(S_{g,n}\backslash D)$ and give rise to transformations of the holonomies along the generating curves m_i, a_j, b_j, thus inducing a map $(G \ltimes \mathfrak{g}^*)^{n+2g} \to (G \ltimes \mathfrak{g}^*)^{n+2g}$ which we denote by Λ. Explicitly, we have

$$\begin{aligned}\Lambda : (\mathrm{Hol}(m_1), \ldots, \mathrm{Hol}(m_n), \mathrm{Hol}(a_1), \mathrm{Hol}(b_1), \ldots, \mathrm{Hol}(a_g), \mathrm{Hol}(b_g))& \\ \mapsto (\mathrm{Hol}(\lambda(m_1)), \ldots, \mathrm{Hol}(\lambda(b_g))).&\end{aligned} \tag{3.2}$$

The push-forward with Λ defines a map $\Lambda_* : \mathcal{C}^\infty((G \ltimes \mathfrak{g}^*)^{n+2g}) \to \mathcal{C}^\infty((G \ltimes \mathfrak{g}^*)^{n+2g})$, which maps the flower algebra into itself. We write Λ_G for the restriction of Λ to the G-components of the holonomies and $(\Lambda_G)_*$ for the push-forward $F \to F \circ \Lambda_G^{-1}$ of functions $F \in \mathcal{C}^\infty(G^{n+2g})$.

In view of Theorem 2.2 it is natural to ask if we can lift the mapping class group action $\lambda \in \mathrm{Map}(S_{g,n}\backslash D) \mapsto \Lambda \in \mathrm{Diff}((G \ltimes \mathfrak{g}^*)^{n+2g})$ to an action $\lambda \in \mathrm{Map}(S_{g,n}\backslash D) \mapsto \tilde{\Lambda} \in \mathrm{Diff}((G \ltimes \mathfrak{g}^*)^{n+2g})$ such that the following diagram commutes

$$\begin{array}{ccc}(G \ltimes \mathfrak{g}^*)^{n+2g} & \xrightarrow{\tilde{\Lambda}} & (G \ltimes \mathfrak{g}^*)^{n+2g} \\ \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \tau} \\ (G \ltimes \mathfrak{g}^*)^{n+2g} & \xrightarrow{\Lambda} & (G \ltimes \mathfrak{g}^*)^{n+2g}.\end{array} \tag{3.3}$$

To define $\tilde{\Lambda}$ note that all generators of the mapping class group, defined by expressions (A.4)-(A.10) and (A.11) in the appendix, either leave the conjugacy classes $\mathcal{C}_{\mu_i s_i}$ associated to each puncture invariant or exchange the conjugacy classes of different punctures. Thus we can perform the following construction.

Let E be a group acting on G^{n+2g} via $\xi \in E \mapsto \Xi_G \in \mathrm{Diff}(G^{n+2g})$, such that E acts on the first n copies of G by conjugation and permutation

$$\begin{aligned} \Xi_G : & u_{M_i} \mapsto \xi_{M_i}(u_{M_1}, \ldots, u_{B_g}) \cdot u_{M_{\sigma(i)}} \cdot \xi_{M_i}^{-1}(u_{M_1}, \ldots, u_{B_g}) \\ & u_{A_j} \mapsto \xi_{A_j}(u_{M_1}, \ldots, u_{B_g}),\ u_{B_j} \mapsto \xi_{B_j}(u_{M_1}, \ldots, u_{B_g}), \end{aligned} \tag{3.4}$$

with maps $\xi_{M_i}, \xi_{A_j}, \xi_{B_j} : G^{n+2g} \to G$ and a permutation $\sigma \in S_n$. Then

$$\begin{aligned} \tilde{\Xi}_G : & v_{M_i} \mapsto \xi_{M_i} \circ \beta(v_{M_1}, \ldots, u_{B_g}) \cdot v_{M_\sigma(i)} \\ & u_{A_j} \mapsto \xi_{A_j} \circ \beta(v_{M_1}, \ldots, u_{B_g}),\ u_{B_j} \mapsto \xi_{B_j} \circ \beta(v_{M_1}, \ldots, u_{B_g}) \end{aligned} \tag{3.5}$$

defines a group action $\xi \in E \to \tilde{\Xi}_G \in \mathrm{Diff}(G^{n+2g})$ and the definition (2.11) of the map β implies

$$\beta \circ \tilde{\Xi}_G = \Xi_G \circ \beta \qquad \forall \xi \in E. \tag{3.6}$$

This allows us to lift the action $\Lambda_G \in \mathrm{Diff}(G^{n+2g})$ of elements $\lambda \in \mathrm{Map}(S_{g,n} \backslash D)$ to an action $\tilde{\Lambda}_G \in \mathrm{Diff}(G^{n+2g})$ related to Λ_G via (3.6). We can extend $\tilde{\Lambda}_G$ to a diffeomorphism $\tilde{\Lambda}$ on $(G \ltimes \mathfrak{g}^*)^{n+2g}$ by taking its action on $(\mathfrak{g}^*)^{n+2g}$ to be the one defined by Λ, which yields a mapping class group action $\lambda \in \mathrm{Map}(S_{g,n} \backslash D) \mapsto \tilde{\Lambda} \in \mathrm{Diff}((G \ltimes \mathfrak{g}^*)^{n+2g})$ satisfying (3.3).

We can then use this lift of the mapping class group action on the flower algebra to an action on $(G \ltimes \mathfrak{g}^*)^{n+2g}$ with symplectic potential (2.13) to prove that the mapping class group action on the flower algebra is a Poisson action:

Theorem 3.1 *(Poisson action of the mapping class group)*

The symplectic potential Θ (2.13) *and the flower algebra Poisson structure* (2.10) *on* $(G \ltimes \mathfrak{g}^*)^{n+2g}$ *are invariant under the mapping class group actions* $\tilde{\Lambda}, \Lambda$, *respectively.*

Proof: For the symplectic potential Θ in (2.13) the invariance under $\tilde{\Lambda}$ can be shown by direct calculation using the expressions (A.4)-(A.11) for the action of the generators of $\mathrm{Map}(S_{g,n} \backslash D)$ on the curves m_i, a_j, b_j, the parametrisation (2.3) and the lift (3.5). The invariance of the flower algebra Poisson structure under λ then follows from Theorem 2.2 and the commutative diagram (3.3), as we have for all $\lambda \in \mathrm{Map}(S_{g,n} \backslash D)$

$$\tau^* \Lambda_* \omega_{\mathcal{F}} = \tilde{\Lambda}_* \tau^* \omega_{\mathcal{F}} = \widetilde{\Lambda}_* d\Theta = d\Theta = \tau^* \omega_{\mathcal{F}}. \tag{3.7}$$

Hence $\Lambda_* \omega_{\mathcal{F}} = \omega_{\mathcal{F}}$ by injectivity of the pullback with the surjective map τ in (2.12). A proof of the invariance of the Poisson bracket $\{\,,\}$ by direct calculation is given in [22] for the case of the (2+1)-dimensional Poincaré group and can easily be extended to the case of a general group $G \ltimes \mathfrak{g}^*$. $\square$

3.2 Infinitesimal generators for the action of Dehn twists

After investigating the mapping class group action on the flower algebra and the Poisson manifold $(G \ltimes \mathfrak{g}^*)^{n+2g}$ with symplectic potential (2.13), we will now demonstrate that the action of Dehn twists can be related to an *infinitesimally generated* Poisson action of the group G.

Let γ be an embedded i. e. non self-intersecting curve on the surface $S_{g,n}\backslash D$ and let the same letter stand for the element of the (pure) mapping class group given by the Dehn twist around γ as outlined in the appendix. Denote by $\Gamma \in \mathrm{Diff}((G \ltimes \mathfrak{g}^*)^{n+2g})$ and $\Gamma_G \in \mathrm{Diff}(G^{n+2g})$ the actions of this Dehn twist on the groups $(G \ltimes \mathfrak{g}^*)^{n+2g}$ and G^{n+2g}, respectively, and by $\tilde{\Gamma} \in \mathrm{Diff}((G \ltimes \mathfrak{g}^*)^{n+2g})$ and $\tilde{\Gamma}_G \in \mathrm{Diff}(G^{n+2g})$ their lifts according to (3.5) and (3.3). Parametrising the holonomy of the curve γ as $\mathrm{Hol}(\gamma) = (u_\gamma, -\mathrm{Ad}^*(u_\gamma^{-1})\boldsymbol{j}^\gamma)$ and expressing it as a product of the holonomies M_i, A_j, B_j, we can introduce coordinate maps j_a^γ analogous to (2.8)

$$j_a^\gamma \in \mathcal{C}^\infty((G \ltimes \mathfrak{g}^*)^{n+2g}) : (M_1, \ldots, M_n, A_1, \ldots, B_g) \mapsto j_a^\gamma, \qquad a = 1, \ldots, \dim G. \tag{3.8}$$

From the brackets (2.10) it follows that the coordinate functions j_a^X generate a G-action on $\mathcal{C}^\infty(G^{n+2g})$ for all generators of the fundamental group. One would like to generalise this statement to any embedded curve γ. This requires one to find G-actions $\rho_\gamma : G \to \mathrm{Diff}(G^{n+2g})$, $\tilde{\rho}_\gamma : G \to \mathrm{Diff}(G^{n+2g})$ on G^{n+2g} that are infinitesimally generated by j_a^γ via these Poisson brackets

$$\{j_a^\gamma, F\} = -\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\, F \circ \rho_\gamma(e^{-tJ^a}) \qquad \forall F \in \mathcal{C}^\infty(G^{n+2g}) \tag{3.9}$$

$$\{j_a^\gamma, F\}_\Theta = -\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\, F \circ \tilde{\rho}_\gamma(e^{-tJ^a}) \qquad \forall F \in \mathcal{C}^\infty(G^{n+2g}). \tag{3.10}$$

Also, these G-actions should act on the group elements associated to the punctures by conjugation and left-multiplication, respectively, such that for each $g \in G$ $\rho_\gamma(g), \tilde{\rho}_\gamma(g) \in \mathrm{Diff}(G^{n+2g})$ are related by (3.5). If such G-actions exist, they are uniquely defined by (3.9), (3.10), since every element of the group G can be written as a product of elements in the image of the exponential map. Remarkably, it is possible to define such G-actions $\rho_\gamma, \tilde{\rho}_\gamma$ for each embedded curve γ on $S_{g,n}\backslash D$, and to relate them to the actions Γ_G, $\tilde{\Gamma}_G$ of the Dehn twist around γ.

Theorem 3.2 *(Action of the Dehn twists on G^{n+2g})*

For any embedded curve γ on the surface $S_{g,n}\backslash D$, Eqs. (3.9) *and* (3.10) *define associated G-actions $\rho_\gamma, \tilde{\rho}_\gamma : G \to \mathit{Diff}(G^{n+2g})$ related by* (3.5)*, which conjugate and, respectively, left-multiply the group elements associated to the punctures. In terms of these group actions, the actions $\Gamma_G, \tilde{\Gamma}_G \in \mathit{Diff}(G^{n+2g})$ of the Dehn twist around γ on G^{n+2g} can be expressed as*

$$\tilde{\Gamma}_G = \tilde{\rho}_\gamma(P_\gamma^{-1} \circ \beta) \qquad \Gamma_G = \rho_\gamma(P_\gamma^{-1}), \tag{3.11}$$

where $P_\gamma^{\pm 1}: G^{n+2g} \to G$, $P_\gamma^{\pm 1}: (u_{M_1}, \ldots, u_{B_g}) \mapsto u_\gamma^{\pm 1}$ *maps to the (inverse of) the* G*-component of* $Hol(\gamma) = (u_\gamma, -Ad^*(u_\gamma^{-1})\boldsymbol{j}^\gamma)$ *expressed as a product in the* G*-components* $u_{M_1}, \ldots, u_{M_n}, u_{A_1}, \ldots, u_{B_g}$.

Proof:

Because of identity (3.6) and the commutative diagram (3.3), it is sufficient to prove the existence of such a G-action for the modified Poisson bracket $\{\,,\,\}_\Theta$ and the action $\tilde{\rho}_\gamma$.

1. As a first step, we show how a G-action $\tilde{\rho}_\gamma$ associated to an embedded curve γ that satisfies (3.10) and (3.11) gives rise to a corresponding group action $\tilde{\rho}_\xi$ for all curves ξ that can be obtained from γ via the action of $\mathrm{Map}(S_{g,n}\backslash D)$. Let $\lambda \in \mathrm{Map}(S_{g,n}\backslash D)$ be an element of the mapping class group with actions $\Lambda_G, \tilde{\Lambda}_G \in \mathrm{Diff}(G^{n+2g})$ on G^{n+2g}. Consider the (embedded) curve ξ obtained from γ by acting with λ, write its holonomy as $\mathrm{Hol}(\xi) = (u_\xi, -\mathrm{Ad}^*(u_\xi^{-1})\boldsymbol{j}^\xi)$ and denote by $\Xi_G, \tilde{\Xi}_G \in \mathrm{Diff}(G^{n+2g})$ the actions of the Dehn twist around ξ on G^{n+2g}. From the geometric definition of Dehn twists in the appendix it follows that the Dehn twists around the curves ξ and γ are related by $\xi = \lambda \circ \gamma \circ \lambda^{-1}$. Using (3.2) and the definition (3.5) of the lifts, we deduce that the associated actions $\tilde{\Gamma}_G$, $\tilde{\Xi}_G$ on G^{n+2g} satisfy $\tilde{\Xi}_G = \tilde{\Lambda}_G^{-1} \circ \tilde{\Gamma}_G \circ \tilde{\Lambda}_G$. On the other hand, the invariance of the symplectic potential Θ (2.13) under $\tilde{\Lambda}$ implies

$$\{j_a^\xi, F\}_\Theta = \{j_a^\gamma, F \circ \tilde{\Lambda}_G^{-1}\}_\Theta \circ \tilde{\Lambda}_G \qquad \forall F \in \mathcal{C}^\infty(G^{n+2g}), \tag{3.12}$$

since $j_a^\xi =_a j^\gamma \circ \tilde{\Lambda}$, and this allows us to define a G-action $\tilde{\rho}_\xi : G \to \mathrm{Diff}(G^{n+2g})$ via

$$\tilde{\rho}_\xi(g) := \tilde{\Lambda}_G^{-1} \circ \tilde{\rho}_\gamma(g) \circ \tilde{\Lambda}_G \qquad \forall g \in G. \tag{3.13}$$

Using the invariance of the Poisson bracket under the mapping class group and the corresponding identity for γ, we see immediately that $\tilde{\rho}_\xi$ satisfies (3.10). Furthermore, since $\tilde{\Lambda}_G$ acts on the group elements associated to the punctures by left-multiplication and permutation, we see that $\tilde{\rho}_\xi$ acts on these elements by left-multiplication if the same is true for $\tilde{\rho}_\gamma$. To prove that $\tilde{\rho}_\xi$ satisfies (3.11), we note that the map P_ξ^{-1} is given by $P_\xi^{-1} = P_\gamma^{-1} \circ \Lambda_G$ and calculate

$$\begin{aligned}\tilde{\rho}_\xi\left(P_\xi^{-1} \circ \beta\right)(v_{M_1}, \ldots u_{B_g}) &= \tilde{\rho}_\xi\left(P_\xi^{-1} \circ \beta(v_{M_1}, \ldots u_{B_g})\right)(v_{M_1}, \ldots u_{B_g}) \\ &= \tilde{\rho}_\xi(P_\gamma^{-1} \circ \Lambda_G \circ \beta(v_{M_1}, \ldots u_{B_g}))(v_{M_1}, \ldots u_{B_g}) \\ &= \tilde{\Lambda}_G^{-1} \circ \tilde{\rho}_\gamma(P_\gamma^{-1} \circ \beta) \circ \tilde{\Lambda}_G(v_{M_1}, \ldots u_{B_g}) \\ &= \tilde{\Lambda}_G^{-1} \circ \tilde{\Gamma}_G \circ \tilde{\Lambda}_G(v_{M_1}, \ldots u_{B_g}) \\ &= \tilde{\Xi}_G(v_{M_1}, \ldots, u_{B_g}).\end{aligned} \tag{3.14}$$

2. We therefore only need to prove (3.11) for a set of curves containing one representative for each orbit of the $\mathrm{Map}(S_{g,n}\backslash D)$-action on $\pi_1(S_{g,n}\backslash D)$. Such a set of curves can be constructed using results from geometric topology [17]. It has been shown, see for example Lemma 2.3.A. in [17], that the equivalence classes of all non-separating curves γ on the

surface $S_{g,n}\backslash D$, i.e. curves γ such that $(S_{g,n}\backslash D)\setminus\gamma$ is connected, are in the same orbit under the action of the mapping class group. This is a consequence of the classification of two-dimensional surfaces via the Euler characteristic. We can apply the same argument to separating curves if we keep in mind that, unlike the handles, the punctures of our surface $S_{g,n}\backslash D$ can be distinguished via the conjugacy classes assigned to them. This allows us to conclude that any two separating curves γ,γ' on $S_{g,n}\backslash D$ such that the two components of $(S_{g,n}\backslash D)\setminus\gamma$ and $(S_{g,n}\backslash D)\setminus\gamma'$ contain the same number of handles and the same sets of punctures lie in the same orbit under the action of the mapping class group. It is therefore sufficient to prove (3.11) for one non-separating curve, for example any of the curves in (A.1) except $\kappa_{\nu,\mu}$, and the separating curves $\gamma^{i_1\ldots i_r j_1\ldots j_s}$ pictured in Fig. 2.

$$\gamma^{i_1\ldots i_r j_1\ldots j_s}=[b_{j_s},a_{j_s}^{-1}]\cdot[b_{j_{s-1}},a_{j_{s-1}}^{-1}]\cdots[b_{j_1},a_{j_1}^{-1}]\cdot m_{i_r}\cdots m_{i_1} \tag{3.15}$$
$$1\leq j_1<j_2<\ldots<j_s\leq j_{s+1}:=g,\ 1\leq i_1<i_2<\ldots<i_r\leq i_{r+1}:=n.$$

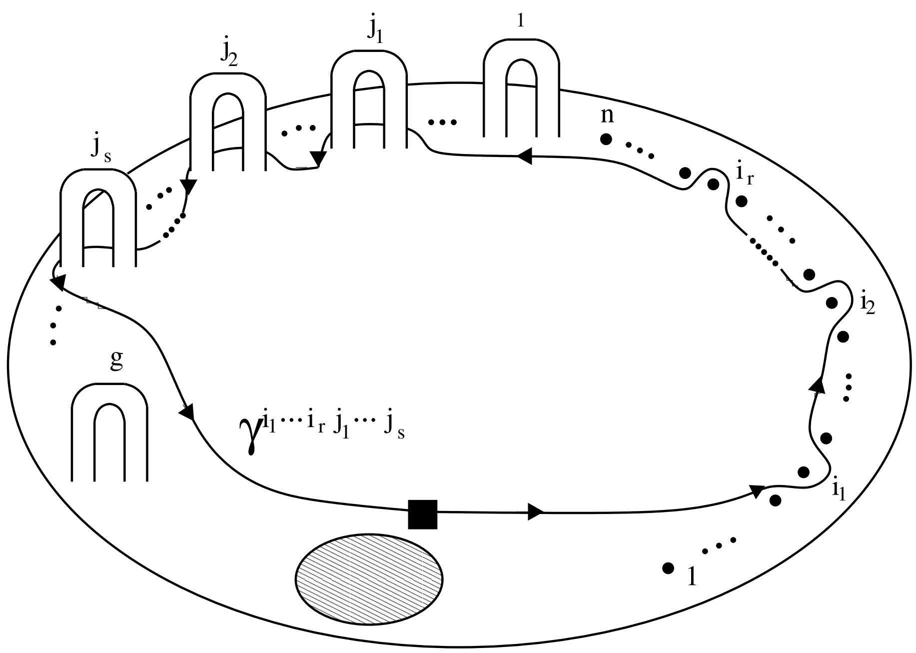

Fig. 2

Separating curves on the surface $S_{g,n}\backslash D$

3. For a given curve γ expressed as a product in the generators $m_i,a_j,b_j\in\pi_1(S_{g,n}\backslash D)$, we can parametrise the holonomy as $\mathrm{Hol}(\gamma)=(u_\gamma,-\mathrm{Ad}^*(u_\gamma^{-1})\boldsymbol{j}^\gamma)$ and express it in terms of the holonomies M_i, A_j, B_j, which allows us to calculate the Poisson brackets $\{j_a^\gamma,F\}_\Theta$ via (2.10), (2.14). For the set of curves (A.1) in the appendix, the holonomies are given

by (A.2), (A.3), and we define the corresponding G-actions as

$$\begin{aligned}
\tilde{\rho}_{a_i}(g) : & u_{A_i} \mapsto g u_{A_i} g^{-1} && (3.16)\\
& u_{B_i} \mapsto [g, u_{A_i}] u_{B_i} g^{-1}\\
& u_X \mapsto [g, u_{A_i}] u_X [g, u_{A_i}]^{-1} \quad \forall X > A_i, B_i\\
\tilde{\rho}_{\delta_i}(g) : & u_{A_i} \mapsto [g, u_{\delta_i}] u_{A_i} g^{-1} && (3.17)\\
& u_{B_i} \mapsto [g, u_{\delta_i}] u_{B_i} [g, u_{\delta_i}]^{-1}\\
& u_X \mapsto [g, u_{\delta_i}] u_X [g, u_{\delta_i}]^{-1} \quad \forall X > A_i, B_i\\
\tilde{\rho}_{\alpha_i}(g) : & u_{A_i} \mapsto [g, u_{\alpha_i}] u_{A_i} g^{-1} && (3.18)\\
& u_{B_i} \mapsto [g, u_{\alpha_i}] u_{B_i} [g, u_{\alpha_i}]^{-1}\\
& u_{A_{i-1}} \mapsto g u_{A_{i-1}}\\
& u_{B_{i-1}} \mapsto g u_{B_{i-1}} g^{-1}\\
& u_X \mapsto [g, u_{\alpha_i}] u_X [g, u_{\alpha_i}]^{-1} \quad \forall X > A_i, B_i\\
\tilde{\rho}_{\epsilon_i}(g) : & u_X \mapsto g u_X g^{-1} \qquad \forall X \in \{A_1, \ldots, B_{i-1}\} && (3.19)\\
& u_{A_i} \mapsto [g, u_{\epsilon_i}] u_{A_i} g^{-1}\\
& u_{B_i} \mapsto [g, u_{\epsilon_i}] u_{A_i} [g, u_{\epsilon_i}]^{-1}\\
& u_X \mapsto [g, u_{\epsilon_i}] u_X [g, u_{\epsilon_i}]^{-1} \qquad \forall X \in \{A_{i+1}, \ldots, B_g\}\\
\tilde{\rho}_{\kappa_{\nu,\mu}}(g) : & v_{M_\nu} \mapsto g v_{M_\nu} && (3.20)\\
& v_{M_\tau} \mapsto [g, u_{M_\nu}] v_{M_\tau} \qquad \forall \nu < \tau < \mu\\
& v_{M_\mu} \mapsto g v_{M_\mu}\\
& v_{M_\tau} \mapsto [g, u_{\kappa_{\nu,\mu}}] v_{M_\tau} \qquad \forall \tau > \mu\\
& u_X \mapsto [g, u_{\kappa_{\nu,\mu}}] u_X [g, u_{\kappa_{\nu,\mu}}]^{-1} \qquad \forall X \in \{A_1, \ldots, B_g\}\\
\tilde{\rho}_{\kappa_{\nu,n+2i-1}}(g) : & v_{M_\nu} \mapsto g v_{M_\nu} && (3.21)\\
& v_{M_\tau} \mapsto [g, u_{M_\nu}] v_{M_\tau} \qquad \forall \tau = \nu+1, \ldots, n,\\
& u_X \mapsto [g, u_{M_\nu}] u_X [g, u_{M_\nu}]^{-1} \qquad \forall X \in \{A_1, \ldots, B_{i-1}\}\\
& u_{A_i} \mapsto [g, u_{\kappa_{\nu,n+2i-1}}] u_{A_i} g^{-1}\\
& u_{B_i} \mapsto [g, u_{\kappa_{\nu,n+2i-1}}] u_{B_i} [g, u_{\kappa_{\nu,n+2i-1}}]^{-1}\\
& u_X \mapsto [g, u_{\kappa_{\nu,n+2i-1}}] u_X [g, u_{\kappa_{\nu,n+2i-1}}]^{-1}\ \forall X \in \{A_{i+1}, \ldots, B_g\}\\
\tilde{\rho}_{\kappa_{\nu,n+2i}}(g) : & v_{M_\nu} \mapsto g v_{M_\nu} && (3.22)\\
& v_{M_\tau} \mapsto [g, u_{M_\nu}] v_{M_\tau} \qquad \forall \tau = \nu+1, \ldots, n\\
& u_X \mapsto [g, u_{M_\nu}] u_X [g, u_{M_\nu}]^{-1} \qquad \forall X \in \{A_1, \ldots, B_{i-1}\}\\
& u_{A_i} \mapsto g u_{A_i} [g, u_{M_\nu}]^{-1}\\
& u_{B_i} \mapsto g u_{B_i} g^{-1}\\
& u_X \mapsto [g, u_{\kappa_{\nu,n+2i}}] u_X [g, u_{\kappa\nu,n+2i}]^{-1} \qquad \forall X \in \{A_{i+1}, \ldots, B_g\},
\end{aligned}$$

where $[\,,\,]$ denotes the group commutator on G as given after (2.13), $M_1 < \ldots < M_n < A_1, B_1 < \ldots < A_g, B_g$, and $(u_{M_1}, \ldots, u_{M_n}, u_{A_1}, \ldots, u_{B_g}) = \beta(v_{M_1}, \ldots, v_{M_n}, u_{A_1}, \ldots, u_{B_g})$. We listed only those elements u_X that transform non-trivially. It can be shown by direct computation that expressions (3.16)-(3.22) define G-actions on G^{n+2g} which satisfy (3.10) and act on the group elements associated to the punctures by left-multiplication. Furthermore, comparing these G-actions with the action $\tilde{\Gamma}_G$ derived from expressions (A.4)-(A.10) in the appendix, we see that they agree if we set $g = u_\gamma^{-1}$.

Similarly, we calculate for the separating curves $\gamma^{i_1\ldots i_r j_1\ldots j_s}$ in (3.15)

$$\begin{aligned}\boldsymbol{j}^{\gamma^{i_1\ldots i_r j_1\ldots j_s}} =& \boldsymbol{j}^{M_{i_1}} + \mathrm{Ad}^*(u_{M_{i_1}})\boldsymbol{j}^{M_{i_2}} + \ldots + \mathrm{Ad}^*(u_{M_{i_r}}\cdots u_{M_{i_1}})\boldsymbol{j}^{H_{j_1}} + \ldots \\ &+ \mathrm{Ad}^*(u_{K_{j_{s-1}}}\cdots u_{K_{j_1}} u_{M_{i_r}}\cdots u_{M_{i_1}})\boldsymbol{j}^{H_{j_s}}\end{aligned} \tag{3.23}$$

with $u_{K_j} = [u_{B_j}, u_{A_j}^{-1}] = u_{B_j} u_{A_j}^{-1} u_{B_j}^{-1} u_{A_j}$ and

$$\boldsymbol{j}^{H_j} = (1 - \mathrm{Ad}^*(u_{A_j}^{-1} u_{B_j}^{-1} u_{A_j}))\boldsymbol{j}^{A_j} + \big(\mathrm{Ad}^*(u_{A_j}^{-1} u_{B_j}^{-1} u_{A_j}) - \mathrm{Ad}^*(u_{B_j}^{-1} u_{A_j})\big)\boldsymbol{j}^{B_j}, \tag{3.24}$$

and define for all $g \in G$

$$\begin{aligned}\tilde{\rho}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}(g) : v_{M_{i_l}} &\mapsto g v_{M_{i_l}} \qquad l = 1,\ldots,r \\ u_{A_{j_l}} &\mapsto g u_{A_{j_l}} g^{-1}, u_{B_{j_l}} \mapsto g u_{B_{j_l}} g^{-1} \qquad l = 1,\ldots,s \\ v_{M_i} &\mapsto [g, u_{M_{i_l}}\cdots u_{M_{i_1}}] v_{M_i} \qquad \forall i_l < i < i_{l+1},\ l = 1,\ldots,r \\ u_{A_j} &\mapsto [g, u_{M_{i_r}}\cdots u_{M_{i_1}}] u_{A_j} [g, u_{M_{i_r}}\cdots u_{M_{i_1}}]^{-1} \qquad \forall 1 \le j < j_1 \\ u_{B_j} &\mapsto [g, u_{M_{i_r}}\cdots u_{M_{i_1}}] u_{B_j} [g, u_{M_{i_r}}\cdots u_{M_{i_1}}]^{-1} \qquad \forall 1 \le j < j_1 \\ u_{A_j} &\mapsto [g, u_{K_{j_l}}\cdots u_{K_{j_1}} u_{M_{i_r}}\cdots u_{M_{i_1}}] u_{A_j} [g, u_{K_{j_l}}\cdots u_{K_{j_1}} u_{M_{i_r}}\cdots u_{M_{i_1}}]^{-1} \\ &\quad \forall j_l < j < j_{l+1},\ l = 1,\ldots,s \\ u_{B_j} &\mapsto [g, u_{K_{j_l}}\cdots u_{K_{j_1}} u_{M_{i_r}}\cdots u_{M_{i_1}}] u_{B_j} [g, u_{K_{j_l}}\cdots u_{K_{j_1}} u_{M_{i_r}}\cdots u_{M_{i_1}}]^{-1} \\ &\quad \forall j_l < j < j_{l+1},\ l = 1,\ldots,s,\end{aligned} \tag{3.25}$$

Again, a straightforward calculation proves that (3.25) defines a G-action on G^{n+2g} which satisfies (3.10) and left-multiplies the group elements associated to the punctures. After determining the action of the Dehn-twists around $\gamma^{i_1\ldots i_r j_1\ldots j_s}$ on the generators of the fundamental group as described in the appendix, we can derive the associated actions $\tilde{\Gamma}_G^{i_1\ldots i_r j_1\ldots j_s} \in \mathrm{Diff}(G^{n+2g})$

$$\begin{aligned}\tilde{\Gamma}_G^{i_1\ldots i_r j_1\ldots j_s} : v_{M_{i_l}} &\mapsto u^{-1}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}} \cdot v_{M_{i_l}} \qquad l = 1,\ldots,r \\ u_{A_{j_l}} &\mapsto u^{-1}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}} \cdot u_{A_{j_l}} \cdot u^{-1}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}} \qquad l = 1,\ldots,s \\ u_{B_{j_l}} &\mapsto u^{-1}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}} \cdot u_{B_{j_l}} \cdot u_{\gamma^{i_1\ldots i_r j_1\ldots j_s}} \qquad l = 1,\ldots,s \\ v_{M_i} &\mapsto [u^{-1}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}, u_{M_{i_l}}\cdots u_{M_{i_1}}] \cdot v_{M_i} \qquad \forall i_l < i < i_{l+1},\ l = 1,\ldots,r \\ u_{A_j} &\mapsto [u^{-1}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}, u_{M_{i_r}}\cdots u_{M_{i_1}}] \cdot u_{A_j} \cdot [u^{-1}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}, u_{M_{i_r}}\cdots u_{M_{i_1}}]^{-1} \\ &\quad \forall 1 \le j < j_1 \\ u_{B_j} &\mapsto [u^{-1}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}, u_{M_{i_r}}\cdots u_{M_{i_1}}] \cdot u_{B_j} \cdot [u^{-1}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}, u_{M_{i_r}}\cdots u_{M_{i_1}}]^{-1} \\ &\quad \forall 1 \le j < j_1\end{aligned} \tag{3.26}$$

$$
\begin{aligned}
u_{A_j} &\mapsto [u^{-1}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}, u_{K_{j_l}}\cdots u_{M_{i_1}}]\cdot u_{A_j}\cdot[u^{-1}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}, u_{K_{j_l}}\cdots u_{M_{i_1}}]^{-1}\\
&\qquad \forall j_l < j < j_{l+1},\ l=1,\ldots,s\\
u_{B_j} &\mapsto [u^{-1}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}, u_{K_{j_l}}\cdots u_{M_{i_1}}]\cdot u_{B_j}\cdot[u^{-1}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}, u_{K_{j_l}}\cdots u_{M_{i_1}}]^{-1}\\
&\qquad \forall j_l < j < j_{l+1},\ l=1,\ldots,s.
\end{aligned}
$$

We then see that they agree with (3.25) if we set $g = u^{-1}_{\gamma^{i_1\ldots i_r j_1\ldots j_s}}$. Therefore, group actions $\tilde\rho_\gamma : G \to \mathrm{Diff}(G^{n+2g})$ satisfying (3.10) and acting on the group elements associated to the punctures by left-multiplication exist for all embedded curves γ on $S_{g,n}\backslash D$ and are related to the action of the associated Dehn twist on G^{n+2g} via (3.11), which was to be shown.□

Theorem 3.2 shows how the actions $\Gamma_G, \tilde\Gamma_G \in \mathrm{Diff}(G^{n+2g})$ arising from the action of a Dehn twists on the holonomies $M_i,\ A_j,\ B_j$ can be related to an action of the group G that is infinitesimally generated by the element j^γ_a via the Poisson brackets $\{\,,\,\}$ and $\{\,,\,\}_\Theta$. This raises the question if the same can be said for the corresponding actions $\Gamma, \tilde\Gamma \in \mathrm{Diff}((G\ltimes\mathfrak{g}^*)^{n+2g})$. We show that this is the case by using the following lemma.

Lemma 3.3 *Let M be a manifold with diffeomorphism group Diff(M) and denote by Vec(M) the space of real vector fields on M. Let $\mathcal{L}$ be the infinite-dimensional Lie algebra $\mathcal{L} =$ Vec$(M)\ltimes \mathcal{C}^\infty(M)$ with Lie bracket*

$$
[X,Y]_{\mathcal{L}} = [X,Y]_{Vec} \qquad [X,F]_{\mathcal{L}} = X.F \qquad [F,G]_{\mathcal{L}} = 0 \tag{3.27}
$$

for $X,Y \in$ Vec(M), $F,G\in\mathcal{C}^\infty(M)$, where $X.F = \frac{d}{dt}|_{t=0}F(h^X_t(m))$ denotes the action of the vector field X on a function $F\in\mathcal{C}^\infty(M)$ with flow h^X_t generated by X. Consider the action of diffeomorphisms $\phi\in$ Diff(M) on functions $F\in\mathcal{C}^\infty(M)$ and vector fields $X\in$ Vec(M) via push-forward

$$
\phi_*F = F\circ\phi^{-1} \qquad (\phi_*X)F = \frac{d}{dt}|_{t=0}F(\phi\circ h^X_t\circ\phi^{-1}). \tag{3.28}
$$

Then, the push-forward with ϕ is a Lie algebra automorphism of $\mathcal{L}$ and any Lie algebra isomorphism $\varphi:\mathcal{L}\mapsto\mathcal{L}$ with $\varphi($Vect$(M))\subset$ Vect(M) and $\varphi|_{\mathcal{C}^\infty(M)} = \phi_*|_{\mathcal{C}^\infty(M)}$ for some $\phi\in$ Diff(M) is equal to the push-forward with ϕ: $\varphi=\phi_*$.

Proof: The first claim states simply the standard properties of the push-forward, $\phi_*(X.F) = (\phi_*X).(\phi_*F)$ and $\phi_*[X,Y]_{Vec} = [\phi_*X,\phi_*Y]_{Vec}$ for $X,Y\in\mathrm{Vec}(M), F\in\mathcal{C}^\infty(M)$, see [1, 20]. The second follows from the fact that a vector field is uniquely determined by its action on functions: $(\varphi X).(\phi_*F) = \varphi(X.F) = \phi_*(X.F) = (\phi_*X).(\phi_*F)$. □

Recalling the definition of the flower algebra and the modified bracket $\{\,,\,\}_\Theta$, we note that the subspace $\mathfrak{g}^{n+2g}\otimes\mathcal{C}^\infty(G^{n+2g})\oplus\mathcal{C}^\infty(G^{n+2g})$ with bracket $\{\,,\,\}_\Theta$ can be viewed as a Lie algebra of type $\mathcal{L}$ in Lemma 3.3. The Poisson brackets of generators j^X_a and functions $F \in \mathcal{C}^\infty(G^{n+2g})$ allow us to identify the former with a basis of the space of vector fields $\mathrm{Vec}(G^{n+2g})$. From the first set of brackets in (2.10) it is then clear that the commutator of two vector fields agrees with the Poisson bracket of the associated elements in $\mathfrak{g}^{n+2g} \otimes \mathcal{C}^\infty(G^{n+2g})$.

For any embedded curve γ the associated Dehn twist acts on G^{n+2g} via the diffeomorphisms $\Gamma_G, \tilde{\Gamma}_G \in \mathrm{Diff}(G^{n+2g})$ that map $\mathrm{Vec}(G^{n+2g})$ to itself and act on functions $F \in \mathcal{C}^\infty(G^{n+2g})$ via the push-forward. We can therefore apply Lemma 3.3 to the mapping class group action on the Lie algebra $\mathrm{Vec}(G^{n+2g}) \ltimes \mathcal{C}^\infty(G^{n+2g})$ with Lie bracket $\{\,,\,\}_\Theta$. As the flower algebra is multiplicatively generated by the coordinate functions j^X_a and functions in $\mathcal{C}^\infty(G^{n+2g})$, this defines the mapping class group action on the flower algebra with bracket $\{\,,\,\}_\Theta$ uniquely, and we see that it is given by the push-forward with $\tilde{\rho}_\gamma(P_\gamma^{-1} \circ \beta)$. Since Γ and $\tilde{\Gamma}$ are related by the commutative diagram (3.3) and ρ_γ, $\tilde{\rho}_\gamma$ by (3.6), the mapping class group action on the flower algebra with bracket $\{\,,\,\}$ is then given by push-forward with $\rho_\gamma(P_\gamma^{-1})$. Recalling from Lemma 3.3 that the push-forwards with $\rho_\gamma(g)$ and $\tilde{\rho}_\gamma(g)$ define a Poisson action of the group G on the flower algebra with bracket $\{\,,\,\}$ and $\{\,,\,\}_\Theta$, we obtain the following theorem.

Theorem 3.4 *(Action of the Dehn twists on the flower algebra)*

1. *For any embedded curve γ on $S_{g,n}\setminus D$ the push-forward with $\rho_\gamma(g), \tilde{\rho}_\gamma(g) \in \mathit{Diff}(G^{n+2g})$ defines a Poisson action of the group G on the flower algebra with bracket $\{\,,\,\}$ and $\{\,,\,\}_\Theta$, respectively.*

2. *The actions Γ, $\tilde{\Gamma}$ of the associated Dehn twist on the flower algebra with bracket $\{\,,\,\}$ and $\{\,,\,\}_\Theta$ are given by the push-forward with $\rho_\gamma(P_\gamma^{-1})$ and $\tilde{\rho}_\gamma(P_\gamma^{-1} \circ \beta)$.*

One might ask if it is possible to define Hamiltonians such that the actions Γ, $\tilde{\Gamma}$ on the flower algebra are realised as their flow for some value of the flow parameter. In the case where the exponential map $\exp : \mathfrak{g} \to G$ is surjective and elements $u \in G$ can be parametrised as $u = \exp(p^a J_a)$ such Hamiltonians can be given explicitly. We can then write the holonomy along the curve γ as $\mathrm{Hol}(\gamma) = (u_\gamma, -\mathrm{Ad}^*(u_\gamma^{-1})\boldsymbol{j}^\gamma)$ with $u_\gamma = \exp(p^a_\gamma J_a)$ and introduce maps

$$p^a_\gamma \in \mathcal{C}^\infty(G^{n+2g}) : (u_{M_1}, \ldots, u_{B_g}) \mapsto p^a_\gamma. \tag{3.29}$$

Considering the algebra element

$$c_\gamma = p^a_\gamma j^\gamma_a \in \mathfrak{g}^{n+2g} \otimes \mathcal{C}^\infty(G^{n+2g}). \tag{3.30}$$

and the one-parameter group of transformations $\phi^\gamma(t), \tilde{\phi}^\gamma(t)$ of the flower algebra generated by c_γ via the Poisson brackets $\{\,,\,\}$ and $\{\,,\,\}_\Theta$

$$\frac{\mathrm{d}}{\mathrm{dt}}|_{t=0}\, \phi^\gamma(t)\chi = \{c_\gamma, \chi\} \qquad \frac{\mathrm{d}}{\mathrm{dt}}|_{t=0}\tilde{\phi}^\gamma(t)\chi = \{c_\gamma, \chi\}_\Theta \tag{3.31}$$
$$\forall \chi \in S\left(\bigoplus_{k=1}^{n+2g} \mathfrak{g}\right) \otimes \mathcal{C}^\infty(G^{n+2g}),$$

we obtain

$$\{c_\gamma, F\} = -p^a_\gamma \frac{\mathrm{d}}{\mathrm{dt}}|_{t=0}\, F\circ\rho_\gamma(e^{-tJ_a}) = \frac{\mathrm{d}}{\mathrm{dt}}|_{t=0}\, F\circ\rho_\gamma(e^{tp^a_\gamma J_a}) \qquad \forall F\in\mathcal{C}^\infty(G^{n+2g}) \quad (3.32)$$

and an analogous expression involving $\{\,,\,\}_\Theta$ and $\tilde\rho_\gamma$. This implies

$$\phi^\gamma(1)F = F\circ\Gamma_G^{-1} \qquad\qquad \tilde\phi^\gamma(1)F = F\circ\tilde\Gamma_G^{-1}. \quad (3.33)$$

Furthermore, it follows from

$$\{c_\gamma, j^X_a\} = p^b_\gamma\{j^\gamma_b, j^X_a\} + j^\gamma_b\{p^b_\gamma, j^X_a\} \qquad \forall X\in\{M_1,\ldots,M_n,A_1,\ldots,B_g\} \quad (3.34)$$

and the structure of the expression for j^γ_b in terms of the coordinate functions j^Y_b, $Y\in\{M_1,\ldots,B_g\}$, associated to the generators of the fundamental group that $\{c_\gamma, j^X_a\}$ is a linear combination these coordinate functions j^Y_b with coefficients in $\mathcal{C}^\infty(G^{n+2g})$. The identification of the coordinate functions j^Y_b with vector fields on G^{n+2g}, discussed after Lemma 3.3, then implies that ϕ^γ maps $\mathrm{Vec}(G^{n+2g})$ to itself.

Theorem 3.5 *The one-parameter groups of transformations $\phi^\gamma(t)$, $\tilde\phi^\gamma(t)$ act as Poisson isomorphisms on the flower algebra with bracket $\{\,,\,\}$ and $\{\,,\,\}_\Theta$, respectively, and agree with the action of the associated Dehn twist at $t=1$*

$$\phi^\gamma(1) = \Gamma_* \qquad\qquad \tilde\phi^\gamma(1) = \tilde\Gamma_*. \quad (3.35)$$

Proof: That the action of $\phi^\gamma(t)$, $\tilde\phi^\gamma(t)$ is a Poisson action $\forall t\in\mathbb{R}$ follows from the fact that they are infinitesimally generated via the Poisson brackets $\{\,,\,\}$ and $\{\,,\,\}_\Theta$ [1, 20]. That they agree with the action of the Dehn twists γ, $\tilde\gamma$ at $t=1$ can be deduced from (3.33), the fact that they are Poisson actions and that they map the space $\mathrm{Vec}(G^{n+2g})$ to itself by using Lemma 3.3. □

4 The quantum action of the mapping class group

In this section we investigate the action of the mapping class group on the quantised flower algebra and its representation spaces defined in Def. 2.3, Def. 2.4. For the case of Chern-Simons theory with compact, semisimple gauge groups the corresponding quantum action of the mapping class group has been studied by Alekseev and Schomerus [5], who claim that elements of the mapping class group act as algebra automorphism on the quantum algebra constructed via their formalism of combinatorial quantisation of Chern-Simons theory[3]. Furthermore, they relate this quantum action of the mapping class group to an action of a quantum group associated to the gauge group of the underlying Chern-Simons theory. In terms of this quantum group action, Dehn twists around embedded curves are given by the action of the ribbon element and the exchange of punctures is implemented via the universal R-matrix.

[3] The result was announced in [5] but the proof does not appear to have been published.

We generalise and prove these results for semidirect product gauge groups of type $G \ltimes \mathfrak{g}^*$. Using the fact that elements of the mapping class group act as Poisson isomorphisms on the classical flower algebra and the rather close relation between classical and quantised flower algebra, we prove that the mapping class group acts on the quantised flower algebra via algebra isomorphisms. We show that this mapping class group action on the quantised flower algebra can be implemented as an action on its representation spaces. Finally, we relate the mapping class group action to representations of a quantum group, which in our case is the quantum double $D(G)$ of the group G.

In the proof of Theorem 4.1. in [23], we demonstrated that any Poisson isomorphism of the flower algebra that maps the subspace $\mathfrak{g}^{n+2g} \otimes \mathcal{C}^\infty(G^{n+2g}) \oplus \mathcal{C}^\infty(G^{n+2g})$ to itself gives rise to an algebra isomorphism of the quantised flower algebra (2.15). As the quantised flower algebra is multiplicatively generated by elements of $\mathcal{C}^\infty(G^{n+2g})$ and $\mathfrak{g}^{n+2g} \otimes \mathcal{C}^\infty(G^{n+2g})$, this algebra isomorphism is uniquely defined by its action on the subspace $\mathfrak{g}^{n+2g} \otimes \mathcal{C}^\infty(G^{n+2g}) \oplus \mathcal{C}^\infty(G^{n+2g})$. Furthermore, if we identify the isomorphic subspaces $\mathfrak{g}^{n+2g} \otimes \mathcal{C}^\infty(G^{n+2g}) \oplus \mathcal{C}^\infty(G^{n+2g})$ in the classical and quantised flower algebra, the quantum action on this subspace agrees with the classical action. Both the generators (A.4)-(A.10) of the pure mapping class group $\mathrm{PMap}(S_{g,n} \setminus D)$ and the generators (A.11) satisfy the condition above, so that the classical action of the mapping class group gives rise to a mapping class group action as algebra automorphisms of the quantised flower algebra. The results in Sect. 3 then allow us to implement this action on the representation spaces in Theorem 2.4. The key observation is that the states $\psi \in V_{\mu_1 s_1 \ldots \mu_n s_n}$ in the Hilbert spaces defined in Def. 2.4 are vector-valued functions on G^{n+2g} satisfying an equivariance condition. Each component of ψ may thus be viewed as an element of the flower algebra with bracket $\{\,,\,\}_\Theta$. This allows us to define a representation of $\mathrm{Map}(S_{g,n} \setminus D)$ on the Hilbert spaces $V_{\mu_1 s_1 \ldots \mu_n s_n}$ by extending its action on the flower algebra componentwise to $\psi \in V_{\mu_1 s_1 \ldots \mu_n s_n}$.

Theorem 4.1 *(Quantum action of the mapping class group)*

1. *Elements $\lambda \in Map(S_{g,n} \setminus D)$ of the mapping class group act as algebra automorphisms $\hat{\Lambda} : \hat{\mathcal{F}} \to \hat{\mathcal{F}}$ on the quantum flower algebra.*

2. *Let $\pi_\lambda \in S_n$ be the permutation associated to λ via the map π in* (3.1) *and*

$$p_\lambda : V_{s_1} \otimes \ldots \otimes V_{s_n} \to V_{s_{\pi_\lambda(1)}} \otimes \ldots \otimes V_{s_{\pi_\lambda(n)}} \tag{4.1}$$

the map which permutes the factors in the tensor product. Then the map

$$L_\lambda : \ \psi \in V_{\mu_1 s_1 \ldots \mu_n s_n} \ \mapsto \ p_\lambda \circ \psi \circ \tilde{\Lambda}_G^{-1} \in V_{\mu_{\pi_\lambda(1)} s_{\pi_\lambda(1)} \ldots \mu_{\pi_\lambda(n)} s_{\pi_\lambda(n)}}, \tag{4.2}$$

defines a representation of the mapping class group on the Hilbert spaces $V_{\mu_1 s_1 \ldots \mu_n s_n}$ given in Theorem 2.4. On the dense subspace $V^\infty_{\mu_1 s_1 \ldots \mu_n s_n}$ carrying the representations of the quantised flower algebra, it satisfies

$$\Pi_{\mu_1 s_1 \ldots \mu_n s_n}(\hat{\Lambda}\chi) = L_\lambda \circ \Pi_{\mu_1 s_1 \ldots \mu_n s_n}(\chi) \circ L_\lambda^{-1} \qquad \forall \chi \in \hat{\mathcal{F}}. \tag{4.3}$$

Proof: The first claim follows from the discussion in [23] as explained above. That (4.2) defines a representation of the mapping class group is a consequence of the properties of the push-forward. To prove identity (4.3), we use the fact that the action of the mapping class group on the quantum flower algebra is uniquely defined by its action on functions $F \in \mathcal{C}^\infty(G^{n+2g})$ and the generators j_a^X, on which it agrees with the corresponding classical action. With expression (2.18) for the action of these elements on the representation spaces $V_{\mu_1 s_1 \ldots \mu_n s_n}$, we obtain

$$\begin{aligned}\Pi_{\mu_1 s_1 \ldots \mu_n s_n}(\hat{\Lambda}(1 \otimes F))L_\lambda \psi &= (((\Lambda_G)_* F) \circ \beta) \cdot L_\lambda \psi = ((\tilde{\Lambda}_G)_*(F \circ \beta)) \cdot L_\lambda \psi \\ &= L_\lambda(\Pi_{\mu_1 s_1 \ldots \mu_n s_n}(1 \otimes F)\psi),\end{aligned} \tag{4.4}$$

where we used the fact that Λ_G and $\tilde{\Lambda}_G$ are related by equation (3.6). Recalling the definition of the action $\tilde{\Lambda}$ on the classical flower algebra and the fact that this action is a Poisson action by Theorem 3.1, we calculate for the action of elements $j_a^X \otimes F \in \mathfrak{g}^{n+2g} \otimes \mathcal{C}^\infty(G^{n+2g})$

$$\begin{aligned}&\Pi_{\mu_1 s_1 \ldots \mu_n s_n}(\hat{\Lambda}(j_a^X \otimes F))L_\lambda \psi = -i\hbar((\Lambda_G)_* F) \circ \beta \cdot \{\tilde{\Lambda}_* j_a^X, L_\lambda \psi\}_\Theta \\ &\quad = -i\hbar(\tilde{\Lambda}_G)_*(F \circ \beta) \cdot L_\lambda \{j_a^X, \psi\}_\Theta = L_\lambda(\Pi_{\mu_1 s_1 \ldots \mu_n s_n}(j_a^X \otimes F)\psi),\end{aligned} \tag{4.5}$$

which together with (4.4) proves the claim. □

We would now like to relate this action of the mapping class group on the Hilbert spaces $V_{\mu_1 s_1 \ldots \mu_n s_n}$ to different representations of a quantum group associated to the gauge group $G \ltimes \mathfrak{g}^*$, generalising the corresponding result for compact, semisimple gauge groups obtained by Alekseev and Schomerus [5]. Whereas the quantum group representations and their relation to elements of the mapping class group are stated rather implicitly there, we find that our formulation admits an explicit constructing relating them to the classical structures discussed in Sect. 3.

The quantum group relevant to our formulation is the quantum double $D(G)$ of the group G. Using the definition given in [23], we can identify the quantum double as a vector space with the space of continuous functions on $G \times G$ with compact support: $D(G) = C_0(G \times G, \mathbb{C})$. In order to exhibit the structure of $D(G)$ as a ribbon-Hopf-*-algebra, it is necessary to introduce Dirac delta functions which are not strictly in $C_0(G \times G, \mathbb{C})$ but can be included by simply adjoining them. Thus we define multiplication $\bullet$, identity 1, co-multiplication Δ, co-unit ϵ, antipode S and involution * via

$$\begin{aligned}(F_1 \bullet F_2)(v,u) &:= \int_G F_1(z,u)\, F_2(z^{-1}v, z^{-1}uz)\, dz \\ 1(v,u) &:= \delta_e(v) \\ (\Delta F)(v_1,u_1;v_2,u_2) &:= F(v_1, u_1 u_2)\, \delta_{v_1}(v_2) \\ \epsilon(F) &:= \int_G F(v,e)\, dv \\ (SF)(v,u) &:= F(v^{-1}, v^{-1}u^{-1}v) \\ F^*(v,u) &:= \overline{F(v^{-1}, v^{-1}uv)}.\end{aligned} \tag{4.6}$$

The universal R-matrix is then given by

$$R(v_1, u_1; v_2, u_2) = \delta_e(v_1)\delta_e(u_1 v_2^{-1}) \tag{4.7}$$

and the central ribbon element c by

$$c(v, u) = \delta_v(u). \tag{4.8}$$

We start by considering Dehn twists around embedded curves γ. In order to relate the action of these Dehn twists to the action of the ribbon element in representations of $D(G)$, we need to find a way of associating such representations of $D(G)$ to each of the curves γ. In view of the classical results in Sect. 3, one could expect these representations to involve the G-action $\tilde{\rho}_\gamma$ and the map $P_\gamma : G^{n+2g} \to G$. To pursue this intuition further, we note that, given an action of the group G on a manifold M together with map $\phi : M \to G$ satisfying a certain compatibility condition, there is a canonical way of constructing representations of the quantum double $D(G)$ on the space $L^2(M)$:

Lemma 4.2 *Let G be a unimodular Lie group with a continuous action $\rho : g \in G \to$ Diff(M) on a manifold M equipped with a Borel measure dm invariant under the G-action ρ. Let $\phi : M \to G$ be a continuous map satisfying the equivariance condition*

$$\phi(\rho(g)m) = g \cdot \phi(m) \cdot g^{-1} \qquad \forall g \in G, m \in M. \tag{4.9}$$

Then a unitary representation $\Pi_{\rho,\Phi}$ of the quantum double $D(G)$ on the space $L^2(M)$ is given by

$$\Pi_{\rho,\phi}(F)\psi\,(m) = \int_G F(z, \phi(m))\psi \circ \rho(z^{-1})(m) dz, \tag{4.10}$$

where dz is the Haar measure on G.

Proof: That (4.10) defines a unitary representation of $D(G)$ can be shown by direct calculation using the definition (4.6) of the quantum double $D(G)$ and the compatibility condition (4.9). □

In our situation, we have $M = G^{n+2g}$ and consider the G-action $\rho = \tilde{\rho}_\gamma$ and the map $\phi = P_\gamma^{-1} \circ \beta$. We need to show that they satisfy (4.9). For the set of curves (A.1) in the appendix and the separating curves (3.15), this follows directly from identity (3.6) and expressions (3.16)-(3.22) and (3.25) for the G-actions. Identity (3.13) then allows us to generalise this result to all embedded curves on $S_{g,n} \backslash D$ as follows. If $\tilde{\rho}_\gamma$ and $P_\gamma^{-1} \circ \beta$ satisfy (4.9) for an embedded curve γ and ξ is obtained from γ via the action of an element $\lambda \in \text{Map}(S_{g,n} \backslash D)$ of the mapping class group, we have

$$\begin{aligned} P_\xi^{-1} \circ \beta \circ \tilde{\rho}_\xi(g) &= P_\gamma^{-1} \circ \Lambda_G \circ \beta \circ \tilde{\Lambda}_G^{-1} \circ \tilde{\rho}_\gamma(g) \circ \tilde{\Lambda}_G \\ &= P_\gamma^{-1} \circ \beta \circ \tilde{\rho}_\gamma(g) \circ \tilde{\Lambda}_G \\ &= g \cdot (P_\gamma^{-1} \circ \beta \circ \tilde{\Lambda}_G) \cdot g^{-1} \\ &= g \cdot (P_\gamma^{-1} \circ \Lambda_G \circ \beta) \cdot g^{-1} \\ &= g \cdot (P_\xi^{-1} \circ \beta) \cdot g^{-1}, \end{aligned} \tag{4.11}$$

so that (4.9) holds for $\tilde{\rho}_\xi$ and $P_\xi^{-1}\circ\beta$ as well. The curves (A.1) and (3.15) are such that, up to homotopy, every embedded curve on $S_{g,n}\backslash D$ can be obtained by acting on one of them with $\mathrm{Map}(S_{g,n}\backslash D)$. Hence the result (4.9) holds for all embedded curves.

By Lemma 4.2, the G-action $\tilde{\rho}_\gamma$ and the map $P_\gamma^{-1}\circ\beta$ then define a unitary representation of the quantum double $D(G)$ on the Hilbert spaces $V_{\mu_1 s_1\ldots\mu_n s_n}$. Using expression (4.8) for the ribbon element, we see that its representation on the space $V_{\mu_1 s_1\ldots\mu_n s_n}$ agrees with the action of the corresponding Dehn twist:

Theorem 4.3 *(Quantum action of Dehn twists)*

1. *For any embedded curve γ on $S_{g,n}\backslash D$ the map*

$$\Pi_\gamma(F)\psi(v_{m_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g}) = \int_G F(z,u_\gamma^{-1})\,\psi\circ\tilde{\rho}_\gamma(z^{-1})(v_{M_1},\ldots,u_{B_g})dz$$
$$= \int_G F\left(z,P_\gamma^{-1}\circ\beta(v_{M_1},\ldots,u_{B_g})\right)\,\psi\circ\tilde{\rho}_\gamma(z^{-1})(v_{M_1},\ldots,u_{B_g})dz \qquad (4.12)$$

   defines a unitary representation of the quantum double $D(G)$ on the Hilbert space $V_{\mu_1 s_1\ldots\mu_n s_n}$.

2. *The action of the central ribbon element $c\in D(G)$ (4.8) on $V_{\mu_1 s_1\ldots\mu_n s_n}$ agrees with the mapping class group action defined in Theorem 4.1*

$$\Pi_\gamma(c)\psi = (\tilde{\Gamma}_G)_*\psi \qquad \forall\psi\in V_{\mu_1 s_1\ldots\mu_n s_n}. \qquad (4.13)$$

To find the representations associated to the generators (A.11) of the braid group, we use the standard result that representations of a quantum group give rise to representations of the braid group via the universal R-matrix [13]. If we associate a representation of the quantum double $D(G)$ to each puncture of the surface $S_{g,n}$ as in [23], the universal R-matrix of $D(G)$ acts on the tensor product of two such representations. The following theorem generalises results of [7, 19].

Theorem 4.4 *(Quantum action of the braid group)*

Define representations $\Pi_{\mu_i s_i}: D(G)\to Hom(V_{\mu_1 s_1\ldots\mu_n s_n}, V_{\mu_1 s_1\ldots\mu_n s_n})$ of $D(G)$ by

$$\Pi_{\mu_i s_i}(F)\psi(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g}) \qquad (4.14)$$
$$= \int_G F(z,v_{M_i}g_{\mu_i}v_{M_i}^{-1})\psi(v_{M_1},\ldots,v_{M_{i-1}},z^{-1}v_{M_i},v_{M_{i+1}},\ldots,u_{B_g})dz,$$

and let $\pi^i: G^{n+2g}\to G^{n+2g}$ be the map that exchanges the i^{th} and $(i+1)^{th}$ copy of G

$$\pi^i:(v_{M_1},\ldots,v_{M_n},u_{A_1},\ldots,u_{B_g})\mapsto(v_{M_1},\ldots,v_{M_{i+1}},v_{M_i},\ldots v_{M_n},u_{A_1},\ldots,u_{B_g}) \qquad (4.15)$$

Then, the action of the generators (A.11) *of the braid group on* $V_{\mu_1 s_1 \ldots \mu_n s_n}$ *is given by*

$$L_{\sigma^i}\psi = p^i((\Pi_{\mu_i s_i} \otimes \Pi_{\mu_{i+1} s_{i+1}})(R)) \circ \pi^i_* \psi \qquad \forall \psi \in V_{\mu_1 s_1 \ldots \mu_n s_n}, \tag{4.16}$$

where $p^i : V_{\mu_1 s_1 \ldots \mu_n s_n} \to V_{s_1} \otimes \ldots \otimes V_{s_{i-1}} \otimes V_{s_{i+1}} \otimes V_{s_i} \otimes V_{s_{i+2}} \otimes \ldots \otimes V_{s_n}$ *exchanges the spaces* V_{s_i} *and* $V_{s_{i+1}}$ *in the tensor product.*

Proof: That (4.14) defines a representation of the quantum double $D(G)$ on $V_{\mu_1 s_1 \ldots \mu_n s_n}$ can be verified by direct calculation using (4.6). To prove (4.16), we insert the definition (4.7) of the universal R-matrix in (4.14) and obtain

$$\begin{aligned}
&p^i(\Pi_{\mu_i s_i} \otimes \Pi_{\mu_{i+1} s_{i+1}})(R)\left(\pi^i_* \Psi\right)(v_{M_1}, \ldots, u_{B_g}) \qquad\qquad (4.17)\\
&\quad = p^i(\Pi_{\mu_i s_i} \otimes \Pi_{\mu_{i+1} s_{i+1}})(R)\psi(v_{M_1}, \ldots, v_{M_{i+1}}, v_{M_i}, \ldots, u_{B_g})\\
&\quad = p^i \int_{G\times G} R(z_1, v_{M_{i+1}} g_{\mu_{i+1}} v^{-1}_{M_{i+1}}, z_2, v_{M_i} g_{\mu_i} v^{-1}_{M_i})\psi(v_{M_1}, \ldots, z_1^{-1} v_{M_{i+1}}, z_2^{-1} v_{M_i}, \ldots, u_{B_g})\\
&\quad = p^i \psi(v_{M_1}, \ldots, v_{M_{i-1}}, v_{M_{i+1}}, (v_{M_{i+1}} g_{\mu_{i+1}} v_{M_{i+1}})^{-1} \cdot v_{M_i}, v_{M_{i+2}}, \ldots, u_{B_g}).
\end{aligned}$$

Recalling the definition of $\tilde{\sigma}^i_G$ via (A.11) and (3.5), we see that this agrees with $L_{\sigma^i}\psi$. $\square$

5 Concluding remarks

In this paper we constructed a Poisson action of the mapping class group $\mathrm{Map}(S_{g,n}\setminus D)$ on the flower algebra and on the representation spaces of the associated quantum algebra. We related the classical action of Dehn twists to an infinitesimally generated G-action and, in the case where the exponential map is surjective, to Hamiltonian flows of certain conjugation invariant functions on $G \ltimes \mathfrak{g}^*$. In the quantum theory, we showed how the mapping class group representation can be expressed in terms of the ribbon element and universal R-matrix of the quantum double $D(G)$. Our results were derived for any connected, simply-connected and unimodular finite-dimensional Lie group G, but the assumptions of connectedness and unimodularity can be dropped at the expense of mild technical complications.

We feel that the mathematical structure of the flower algebra makes it an object of investigation in its own right. However, it attracted our attention because of its relevance to physics, more precisely, its role in the description of the phase space of (2+1)-dimensional gravity in the Chern-Simons formulation. The phase space of Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$, the moduli space of flat $G \ltimes \mathfrak{g}^*$-connections, can be obtained from our set of holonomies $M_1, \ldots, M_n, A_1, B_1, \ldots A_g, B_g$ by imposing the condition (2.6) and dividing by the $G \ltimes \mathfrak{g}^*$-action which simultaneously conjugates all holonomies. Formal arguments, based on the analogy with the discussion in [5], suggest that the Poisson action of $\mathrm{Map}(S_{g,n}\setminus D)$ on the flower algebra descends to a symplectic action of $\mathrm{Map}(S_{g,n})$ on the moduli space. A mathematically rigorous implementation of these arguments for the non-compact groups $G \ltimes \mathfrak{g}^*$ considered here would be interesting, particularly for the physically relevant cases where $G \ltimes \mathfrak{g}^*$ is the universal cover of the three-dimensional Euclidean or Poincaré group.

In the quantum theory, the classical conjugation symmetry is replaced by an action of the quantum double $D(G)$ on the Hilbert spaces $V_{\mu_1 s_1 \ldots \mu_n s_n}$, see [22], in particular Eq. (4.27). The constraint (2.6) is then implemented on these Hilbert spaces by imposing invariance under the action of the quantum double. The action of $\mathrm{Map}(S_{g,n} \backslash D)$ on $V_{\mu_1 s_1 \ldots \mu_n s_n}$ derived in Sect. 4 of the present paper commutes with this action of the quantum double. Formally, one obtains an action of the mapping class group $\mathrm{Map}(S_{g,n})$ on the invariant states in $V_{\mu_1 s_1 \ldots \mu_n s_n}$, but there are again technical difficulties related to the non-compactness of $G \ltimes \mathfrak{g}^*$: the states which are invariant under the $D(G)$-action are singular and not proper elements of $V_{\mu_1 s_1 \ldots \mu_n s_n}$. For the case of Chern-Simons theory with the non-compact but semisimple gauge group $SL(2, \mathbb{C})$, a mathematically rigorous way of defining invariant states has been derived in [11]. It would be interesting to see if a similar method can be applied to our situation and to investigate the resulting mapping class group action on the reduced Hilbert space, again with particular attention to the three-dimensional Euclidean or Poincaré group.

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A The generators of the mapping class group

For the convenience of the reader we give a set of generators of the mapping class groups $\mathrm{Map}(S_{g,n})$ and $\mathrm{Map}(S_{g,n}\backslash D)$ with explicit formulae for their actions on a set of generators of the fundamental groups $\pi_1(S_{g,n})$ and $\pi_1(S_{g,n}\backslash D)$. Some of the contents of this appendix are taken from our discussion in [22].

The pure mapping class group $\mathrm{PMap}(S_{g,n}\backslash D)$ of the punctured surface $S_{g,n}\backslash D$ is the subgroup of $\mathrm{Map}(S_{g,n}\backslash D)$ which leaves each puncture fixed. A set of generators and defining relations has been derived by Birman [9, 10] but for us the set of generators used by Schomerus and Alekseev [5] is more convenient, which was first given in [24], see also [21]. Note that this set also generates $\mathrm{PMap}(S_{g,n})$, with additional relations.

The generating set consists of Dehn twists around the curves a_i, δ_i, α_i, ϵ_i, $\kappa_{\nu,\mu}$, $\kappa_{\nu,n+2i-1}$ and $\kappa_{\nu,n+2i}$ pictured in Fig. 3.

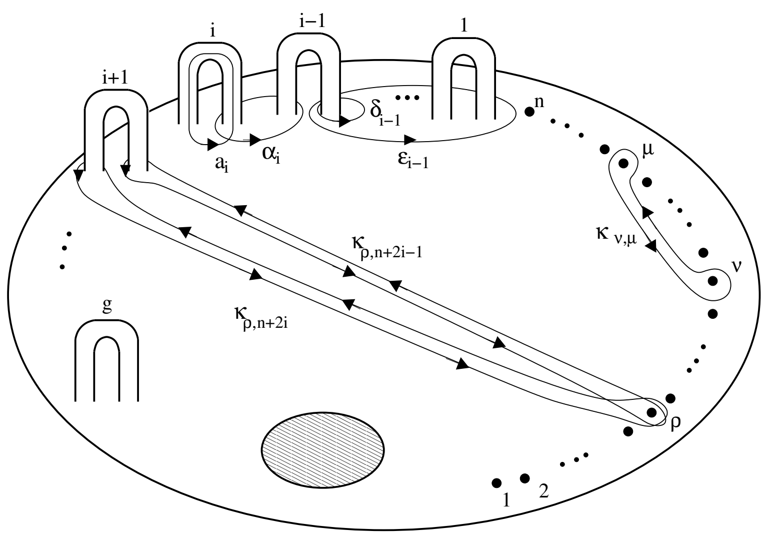

Fig. 3

The curves associated to the generators of the (pure) mapping class group $\mathrm{PMap}(S_{g,n}\backslash D)$

They can be expressed in terms of the generators $m_1, \ldots, m_n, a_1, b_1, \ldots, a_g, b_g$ of the fundamental group shown in Fig. 1 as follows:

$$\begin{aligned}
&a_i && i = 1, \ldots, g \qquad \text{(A.1)}\\
&\delta_i = a_i^{-1} b_i^{-1} a_i && i = 1, \ldots, g\\
&\alpha_i = a_i^{-1} b_i^{-1} a_i b_{i-1} && i = 2, \ldots, g\\
&\epsilon_i = a_i^{-1} b_i^{-1} a_i \cdot (b_{i-1} a_{i-1}^{-1} b_{i-1}^{-1} a_{i-1}) \cdot \ldots \cdot (b_1 a_1^{-1} b_1^{-1} a_1) && i = 2, \ldots, n\\
&\kappa_{\nu,\mu} = m_\mu m_\nu && 1 \leq \nu < \mu \leq n\\
&\kappa_{\nu,n+2i-1} = a_i^{-1} b_i^{-1} a_i m_\nu && \nu = 1, \ldots, n,\ i = 1, \ldots, g\\
&\kappa_{\nu,n+2i} = b_i m_\nu && \nu = 1, \ldots, n,\ i = 1, \ldots, g\ .
\end{aligned}$$

Parametrising the corresponding holonomies as $\mathrm{Hol}(\gamma)=(u_\gamma,-\mathrm{Ad}^*(u_\gamma^{-1})\boldsymbol{j}^\gamma)$ and expressing them in terms of holonomies of the generators m_i, a_j, b_j, we obtain

$$
\begin{aligned}
&u_{a_i} = u_{A_i} \\
&u_{\delta_i} = u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i} \\
&u_{\alpha_i} = u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}u_{B_{i-1}} \\
&u_{\epsilon_i} = u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}u_{K_{i-1}}\cdots u_{K_1} \\
&u_{\kappa_{\nu,\mu}} = u_{M_\mu}u_{M_\nu} \\
&u_{\kappa_{\nu,n+2i-1}} = u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}u_{M_\nu} \\
&u_{\kappa_{\nu,n+2i}} = u_{B_i}u_{M_\nu}
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
&\boldsymbol{j}_{a_i} = \boldsymbol{j}_{A_i} \\
&\boldsymbol{j}_{\delta_i} = (1-\mathrm{Ad}^*(u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}))\boldsymbol{j}_{A_i} - \mathrm{Ad}^*(u_{B_i}^{-1}u_{A_i})\boldsymbol{j}_{B_i} \\
&\boldsymbol{j}_{\alpha_i} = \boldsymbol{j}_{B_{i-1}} + (\mathrm{Ad}^*(u_{B_{i-1}}) - \mathrm{Ad}^*(u_{\alpha_i}))\boldsymbol{j}_{A_i} - \mathrm{Ad}^*(u_{B_i}^{-1}u_{A_i}u_{B_{i-1}})\boldsymbol{j}_{B_i} \\
&\boldsymbol{j}_{\epsilon_i} = \sum_{l=1}^{i-1}\mathrm{Ad}^*(u_{K_{l-1}}\cdots u_{K_1})\boldsymbol{j}_{H_l} + (\ \mathrm{Ad}^*(u_{K_{i-1}}\cdots u_{K_1}) - \mathrm{Ad}^*(u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}u_{K_{l-1}}\cdots u_{K_1})\ )\boldsymbol{j}_{A_i} \\
&\qquad - \mathrm{Ad}^*(u_{B_i}^{-1}u_{A_i}u_{K_{l-1}}\cdots u_{K_1})\ )\boldsymbol{j}_{B_i} \\
&\boldsymbol{j}_{\kappa_{\nu,\mu}} = \boldsymbol{j}_{M_\nu} + \mathrm{Ad}^*(u_{M_\nu})\boldsymbol{j}_\mu \\
&\boldsymbol{j}_{\kappa_{\nu,n+2i-1}} = \boldsymbol{j}_{M_\nu} + (\mathrm{Ad}^*(u_{M_\nu}) - \mathrm{Ad}^*(u_{\kappa_{\nu,n+2i-1}}))\boldsymbol{j}_{A_i} - \mathrm{Ad}^*(u_{A_i}u_{\kappa_{\nu,n+2i-1}})\boldsymbol{j}_{B_i} \\
&\boldsymbol{j}_{\kappa_{\nu,n+2i}} = \boldsymbol{j}_{u_{M_\nu}} + \mathrm{Ad}^*(u_{M_\nu})\boldsymbol{j}_{B_i},
\end{aligned}
\tag{A.3}
$$

with $u_{K_i}=[u_{B_i},u_{A_i}^{-1}]=u_{B_i}u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}$ and $\boldsymbol{j}^{H_i}$ given by (3.24).

A Dehn twist around an embedded curve γ can be defined by embedding a small annulus around the curve and twisting its ends by an angle of 2π as shown in Fig. 4. It induces an outer automorphism of the fundamental group, affecting only those elements for which all representing curves intersect with γ. If we choose a representing curve that has the smallest possible number of intersections with γ for each of the generators m_i, a_j, b_j of the fundamental group and determine their transformations by drawing the images of these curves as indicated in Fig. 4, we obtain explicit formulae for the action of the pure mapping class group on these generators. We summarise this action in the following table, listing only the generators that do not transform trivially[4].

$$
a_i : b_i \mapsto b_i a_i \tag{A.4}
$$

$$
\delta_i : a_i \mapsto a_i\delta_i = b_i^{-1}a_i \tag{A.5}
$$

[4]Note that the Dehn twist (A.5) is the inverse of the twist in [22].

$$
\begin{aligned}
\alpha_i : & a_i \mapsto b_i^{-1} a_i b_{i-1} = a_i \alpha_i \\
& b_{i-1} \mapsto b_{i-1}^{-1} a_i^{-1} b_i a_i b_{i-1} a_i^{-1} b_i^{-1} a_i b_{i-1} = \alpha_i^{-1} b_{i-1} \alpha_i \\
& a_{i-1} \mapsto b_{i-1}^{-1} a_i^{-1} b_i a_i a_{i-1} = \alpha_i^{-1} a_{i-1}
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
\epsilon_i : & a_i \mapsto b_i^{-1} a_i k_{i-1} \dots k_1 = a_i \epsilon_i \\
& a_k \mapsto k_1^{-1} \dots k_{i-1}^{-1} (a_i^{-1} b_i a_i) a_k (a_i^{-1} b_i^{-1} a_i) k_{i-1} \dots k_1 = \epsilon_i^{-1} a_k \epsilon_i \\
& \qquad \forall 1 \leq k < i \\
& b_k \mapsto k_1^{-1} \dots k_{i-1}^{-1} (a_i^{-1} b_i a_i) b_k (a_i^{-1} b_i^{-1} a_i) k_{i-1} \dots k_1 = \epsilon_i^{-1} b_k \epsilon_i \\
& \qquad \forall 1 \leq k < i
\end{aligned}
\tag{A.7}
$$

where $k_j := b_j a_j^{-1} b_j^{-1} a_j$

$$
\begin{aligned}
\kappa_{\nu,\mu} : & m_\nu \mapsto m_\nu^{-1} m_\mu^{-1} m_\nu m_\mu m_\nu = \kappa_{\nu,\mu}^{-1} m_\nu \kappa_{\nu,\mu} \\
& m_\mu \mapsto m_\nu^{-1} m_\mu m_\nu = \kappa_{\nu,\mu}^{-1} m_\mu \kappa_{\nu,\mu} \\
& m_\kappa \mapsto m_\nu^{-1} m_\mu^{-1} m_\nu m_\mu m_\kappa m_\mu^{-1} m_\nu^{-1} m_\mu m_\nu \qquad \forall \nu < \kappa < \mu
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
\kappa_{\nu,n+2i-1} : & m_\nu \mapsto m_\nu^{-1} a_i^{-1} b_i a_i m_\nu a_i^{-1} b_i^{-1} a_i m_\nu = \kappa_{\nu,n+2i-1}^{-1} m_\nu \kappa_{\nu,n+2i-1} \\
& a_i \mapsto b_i^{-1} a_i m_\nu = a_i \kappa_{\nu,n+2i-1} \\
& x_j \mapsto m_\nu^{-1} a_i^{-1} b_i a_i m_\nu a_i^{-1} b_i^{-1} a_i x_j a_i^{-1} b_i a_i m_\nu^{-1} a_i^{-1} b_i^{-1} a_i m_\nu \\
& \quad = \kappa_{\nu,n+2i-1}^{-1} m_\nu \kappa_{\nu,n+2i-1} m_\nu^{-1} x_j m_\nu \kappa_{\nu,n+2i-1}^{-1} m_\nu^{-1} \kappa_{\nu,n+2i-1}, \\
& \qquad x_j \in \{m_{\nu+1}, \dots, m_n, a_1, \dots, b_{i-1}\}
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
\kappa_{\nu,n+2i} : & m_\nu \mapsto m_\nu^{-1} b_i^{-1} m_\nu b_i m_\nu = \kappa_{\nu,n+2i}^{-1} m_\nu \kappa_{\nu,n+2i} \\
& b_i \mapsto m_\nu^{-1} b_i m_\nu = \kappa_{\nu,n+2i}^{-1} b_i \kappa_{\nu,n+2i} \\
& a_i \mapsto m_\nu^{-1} b_i^{-1} a_i b_i^{-1} m_\nu^{-1} b_i m_\nu \\
& \quad = \kappa_{\nu,n+2i}^{-1} a_i b_i^{-1} m_\nu^{-1} \kappa_{\nu,n+2i} \\
& x_j \mapsto m_\nu^{-1} b_i^{-1} m_\nu b_i x_j b_i^{-1} m_\nu^{-1} b_i m_\nu \\
& \quad = \kappa_{\nu,n+2i}^{-1} m_\nu b_i x_j b_i^{-1} m_\nu^{-1} \kappa_{\nu,n+2i}, \qquad x_j \in \{m_{\nu+1}, \dots, m_n, a_1, \dots b_{i-1}\}.
\end{aligned}
\tag{A.10}
$$

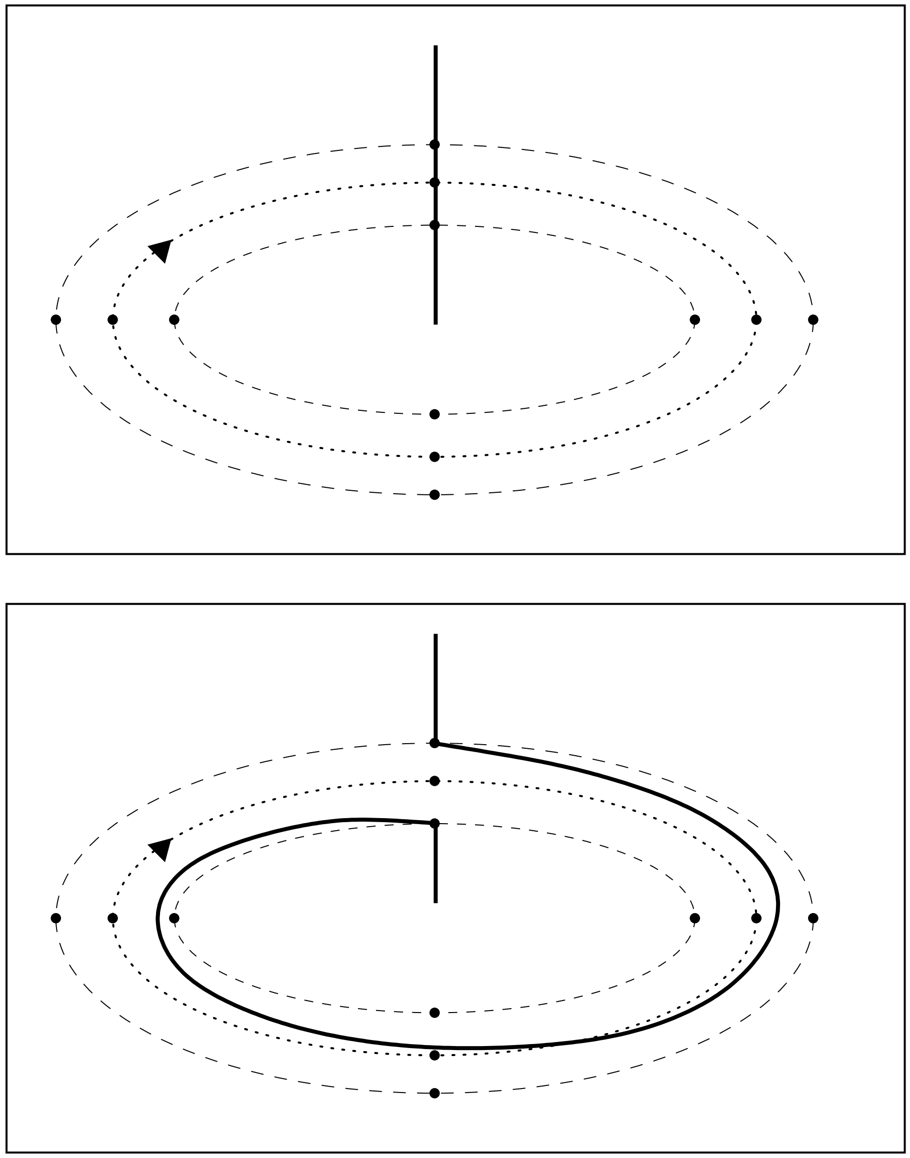

Fig. 4

The effect of a Dehn twist around an oriented loop (dotted line) on a curve intersecting the loop transversally (full line)

A set of generators of the full mapping class group of the surface $S_{g,n}\backslash D$ is obtained by supplementing this set of generators with the generators σ^i, $i = 1, \ldots, n$ of the braid group. The action of these generators on the loops m_i around the punctures is shown in Fig. 5. They leave invariant all generators of the fundamental group except m_i and m_{i+1}, on which they act according to

$$\begin{aligned} \sigma^i : & m_i \mapsto m_{i+1} \\ & m_{i+1} \mapsto m_{i+1} m_i m_{i+1}^{-1} \, . \end{aligned} \tag{A.11}$$

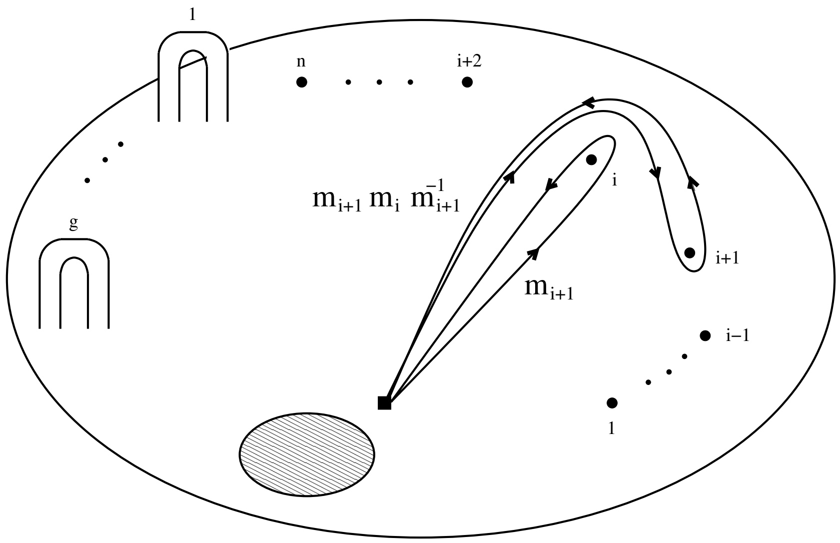

Fig. 5

The generators of the braid group on the surface $S_{g,n}\backslash D$

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