MS-00-007
math.QA/0006228

Combinatorial quantisation of Euclidean gravity in three dimensions

B.J. Schroers[1]
Department of Mathematics and Statistics, University of Edinburgh
King's Buildings, Mayfield Road, Edinburgh EH9 3JZ
United Kingdom
bernd@maths.ed.ac.uk

10 June 2000

Abstract

In the Chern-Simons formulation of Einstein gravity in 2+1 dimensions the phase space of gravity is the moduli space of flat G-connections, where G is a typically non-compact Lie group which depends on the signature of space-time and the cosmological constant. For Euclidean signature and vanishing cosmological constant, G is the three-dimensional Euclidean group. For this case the Poisson structure of the moduli space is given explicitly in terms of a classical r-matrix. It is shown that the quantum R-matrix of the quantum double $D(SU(2))$ provides a quantisation of that Poisson structure.

MSC 17B37, 81R50, 81S10, 83C45

1 Introduction

The primary goal of this paper is to indicate how some of the quantisation techniques developed in the quantisation of Chern-Simons theory with a compact gauge group can be extended and applied to the quantisation of three-dimensional gravity. One expects this to be possible because gravity in three dimensions can be re-formulated as a Chern-Simons theory. The gauge group of the Chern-Simons theory, however, depends on the cosmological constant and the signature of space-time and is non-compact in almost all cases. Here we shall follow the combinatorial quantisation program developed for

[1]address after 1.09.2000: Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom

Chern-Simons theory with a compact gauge group by Alekseev, Grosse and Schomerus, see [1][2] and also [3]. We shall show how to implement the main steps of that program in one particular case, namely Euclidean gravity without cosmological constant. While our results provide key stepping stones on a promising path to a quantisation of three dimensional gravity, a number of issues - both physical and mathematical - are only raised but not settled here. A secondary purpose of this paper is to advertise these issues to mathematicians and physicists with an active interest in the geometry and quantisation of moduli spaces of flat connections.

The paper naturally falls into two halves. The first half, consisting of sects. 2 and 3, is a review of the Chern-Simons formulation of three-dimensional gravity. The precise relation between this formulation and the original Einstein formulation of gravity is a bone of contention in the literature. We indicate some of the issues but do not enter deeply into their discussion. Our view is that the Chern-Simons formulation offers a promising avenue towards quantising three-dimensional gravity, which is worth pursuing because of the significance of the goal and the possibility of concrete results. At the end of sect. 3 the problem of quantising three-dimensional gravity will have been translated into the problem of quantising certain moduli spaces.

The new results of this paper are contained in sects. 4 and 5, which deal with the quantisation problem for Euclidean gravity with vanishing cosmological constant. In that case the gauge group of the Chern-Simons formulation is the (double cover of the) Euclidean group $ISO(3)$ in three dimensions. If space-time is a direct product of time and a two-dimensional space Σ, the phase space of gravity can be identified with the space of flat $ISO(3)$ bundles on Σ. In the framework of combinatorial quantisation, the starting point of the quantisation is the Fock-Rosly description of the Poisson structure of the phase space in terms of a classical r-matrix, i.e. a solution of the classical Yang-Baxter equation [4]. The key step in the quantisation of the phase space is the identification of a solution of the quantum Yang-Baxter equations which reduces to the classical r-matrix in a suitable limit. Here we shall give the r-matrix relevant for $ISO(3)$ Chern-Simons theory and show that it is the limit of the universal R-matrix of the quantum double $D(SU(2))$. The pivotal role of the quantum double $D(SU(2))$ in the quantisation of $ISO(3)$ Chern-Simons theory was discoverd by Bais and Muller in [5]. In our sect. 5 we shall show that $D(SU(2))$ is a deformation of the group algebra of $ISO(3)$ and thereby resolve a question posed in [5].

In this paper we restrict attention to Euclidean gravity and vanishing cosmological constant. The physically more interesting case of Lorentzian signature with vanishing cosmological constant can be dealt with in an analogous manner. However, some additional technical problems and numerous physical implications call for a more detailed discussion, which we give in a separate paper [6]. The inclusion of a non-vanishing cosmological constant in the combinatorial quantisation of three-dimensional gravity poses a very interesting problem. This problem is addressed in [7], and we will briefly comment on it from our viewpoint at the end of this paper.

2 The Chern-Simons formulation of gravity in three dimensions

The possibility of writing general relativity in three dimensions as a Chern-Simons theory was first noticed in [8]. This observation opened up a new approach to gravity and in particular to its quantisation, which was first systematically explored in [9]. Since then, a vast body of literature has been devoted to the subject. This section is an attempt to give the briefest possible summary of the Chern-Simons formulation of three-dimensional Euclidean gravity. For more background and references on three-dimensional gravity we refer the reader to the recent book by Carlip [10] or the review article [11].

In three dimensional gravity, space-time is a three-dimensional manifold M. In the following we shall only consider space times of the form $M = \mathbb{R} \times \Sigma$, where Σ is an orientable two-dimensional manifold ("space"). A three-manifold of that form is orientable and hence, by a classic theorem of Stiefel, parallelisable. Thus its tangent bundle is topologically trivial.

In Einstein's original formulation of general relativity, the dynamical variable is a metric g on M. For our purposes it is essential to adopt Cartan's point of view, where the theory is formulated as a gauge theory. In this approach one introduces an auxiliary 3-dimensional vector bundle V with an inner product $(\ ,\ )$ and connection ω, metric with respect to $(\ ,\ )$. The topological type of V is that of the tangent bundle TM of M (i.e. trivial in our case) and the structure group of V is $SO(3)$ (in the Lorentzian case it would be the Lorentz group $SO(2,1)$). Then there exists a bundle map $TM \to V$ covering the identity. Such a bundle map provides an identification of T_xM with the fibre V_x of V over $x \in M$, and can be thought of as a V-valued one-form (soldering form or dreibein) e on M. Choosing a basis $\{E_a\}$, $a = 1, 2, 3$, of V_x, orthonormal with respect to $(\ ,\ )$ and local coordinates x_μ, $\mu = 1, 2, 3$, around x, we require that the 3×3 matrix $((e^a_\mu))$ defined by $e(\partial_\mu) = \sum_{a=1}^{3} e^a_\mu E_a$ is invertible.

To continue, we introduce generators J_a of the Lie algebra $so(3)$. They are normalised to satisfy

$$[J_a, J_b] = \epsilon_{abc} J_c, \tag{2.1}$$

where ϵ_{abc} is the totally antisymmetric tensor in three dimensions, normalised so that $\epsilon_{123} = 1$. Here and in the following, repeated indices are summed on; since we are in the Euclidean situation, the position of indices (upstairs or downstairs) is purely for notational convenience. The connection one-form ω can be expanded as $\omega = \omega_a J^a$. Similarly the curvature two-form $F_\omega = d\omega + \frac{1}{2}[\omega, \omega]$ can be expanded as $F_\omega = F^a_\omega J_a$, with

$$F^a_\omega = d\omega^a + \frac{1}{2}\epsilon^a_{\ bc}\omega^b \wedge \omega^c. \tag{2.2}$$

The Einstein-Hilbert action in three dimension can be written as

$$S_{EH}[\omega, e] = \int_M e_a \wedge F^a_\omega. \tag{2.3}$$

In Cartan's formulation, both the connection ω and the dreibein e should be thought of as dynamical variables and varied independently. Variation with respect to ω yields the requirement that the connection ω has vanishing torsion:

$$D_\omega e_a = de_a + \frac{1}{2}\epsilon_{abc}\omega^b e^c = 0. \tag{2.4}$$

This condition, which is imposed *a priori* in Einstein's formulation of general relativity, is thus seen to be part of the equation of motion in Cartan's formulation. Variation with respect to e yields the vanishing of the curvature tensor:

$$F_\omega = 0. \tag{2.5}$$

In three dimensions this is equivalent to the vanishing of the Ricci tensor, and thus to Einstein equations in the absence of matter.

The geometrical background to Cartan's formulation is explained beautifully in the book [12]. To explain the key ideas in the present context we choose a global frame E_a and consider globally defined soldering forms e_a on M. A Cartan connection may then be defined as a one-form with values in the Lie algebra $iso(3)$ of the Euclidean group $ISO(3) = \mathbb{R}^3 \rtimes SO(3)$. Thus, if we introduce translation generators P_a, $a = 1, 2, 3$, which satisfy

$$[J_a, P_b] = \epsilon_{abc}P^c, \quad [P_a, P_b] = 0, \tag{2.6}$$

the Cartan connection can be written as

$$A = \omega_a J^a + e_a P^a. \tag{2.7}$$

The Cartan connection should be contrasted with the usual Ehresmann notion of a connection on a principal fibre bundle. While the Ehresmann connection is a one-form with values in the Lie-algebra of the structure group, the Cartan connection takes values in a bigger Lie algebra. In the present case, it is a connection on a principal $SO(3)$ bundle, but takes values in $iso(3)$. Cartan connections also have to satisfy a non-degeneracy condition, which in the present context requires that the soldering forms e_a are nowhere vanishing. Finally we note that the curvature of the Cartan connection

$$F = (D_\omega e^a)P_a + (F^a_\omega)J_a \tag{2.8}$$

combines the curvature and the torsion of the spin connection.

The Cartan framework allows one to translate Riemannian (or Lorentzian) geometry into an equivalent gauge theory. However, the resulting gauge theory is not of the standard type, and conditions have to be imposed on the connection. The crucial - and contentious - step in rewriting writing three dimensional gravity as a Chern-Simons theory is to drop these conditions and to interpret (2.7) as an Ehresmann connection of a bundle whose structure group is $ISO(3)$. Thus, in particular the condition of the invertibility of e^a_μ is dropped. We will not enter into the discussion of the merits and drawbacks of this approach. The advantages for the quantisation are explained in Witten's original paper [9]. For a recent, carefully argued criticism, see [13].

A final technical ingredient we need in order to establish the Chern-Simons formulation is special to three dimensional space-times. This is a non-degenerate, invariant bilinear form on the Lie algebra $iso(3) = \mathbb{R}^3 \rtimes so(3)$:

$$\langle J_a, P_a\rangle = \delta_{ab}, \quad \langle J_a, J_b\rangle = \langle P_a, P_b\rangle = 0. \tag{2.9}$$

Note that with respect to this inner product both $so(3)$ and $\mathbb{R}^3$ are maximally and totally null.

Finally, we can write down the Chern-Simons action on for the connection A on $M = \Sigma \times \mathbb{R}$:

$$S_{CS}[A] = \frac{1}{2}\int_M \langle A \wedge dA\rangle + \frac{2}{3}\langle A \wedge A \wedge A\rangle. \tag{2.10}$$

A short calculation shows that this is equal to the Einstein-Hilbert action (2.3). Moreover, the equation of motion found by varying the action with respect to A is

$$F = 0. \tag{2.11}$$

Using the decomposition (2.8) we thus reproduce the condition of vanishing torsion and the three dimensional Einstein equations, as required.

So far we have only studied the Einstein equations in vacuum. The introduction of matter in the form of point particles is physically desirable. Happily, it can be implemented in a mathematically elegant fashion in the Chern-Simons formulation. We refer the reader to [5] for a detailed discussion and further references, and only summarise the salient points here. Particles are introduced by marking points on the surface Σ and coupling the particle's phase space to the phase space of the theory. The phase space of a particle with Euclidean mass μ and spin s is a co-adjoint orbit $\mathcal{O}_{\mu s}$ of $ISO(3)$. To describe these orbits we write P_a^* and J_a^* for the basis elements of $iso(3)^*$ dual to P_a and J_a, and we write an element $\xi^* \in iso(3)^*$ as

$$\xi^* = p^a P_a^* + j^a J_a^*. \tag{2.12}$$

Using the inner product (2.9) we can identify ξ^* with the element

$$\xi = p^a J_a + j^a P_a \tag{2.13}$$

in $iso(3)$. Then p^a should be thought of as the energy-momentum vector of the particle and j^a as its generalised angular momentum. The orbit $\mathcal{O}_{\mu s}$ consists of all $\xi^\star \in iso(3)^*$ satisfying the mass-shell condition $p_a p^a = \mu^2$ and the spin condition $p_a j^a = \mu s$. As explained in detail in the book [14], $\mathcal{O}_{00}$ is a point, $\mathcal{O}_{0s}$, with $s \neq 0$, is a two-sphere of radius s and $\mathcal{O}_{\mu s}$ for $\mu \neq 0$ is diffeomorphic to TS^2. Co-adjoin torbits have a canonical symplectic structure, often called the Kostant-Kirillov symplectic structure. For the generic case $\mu \neq 0$ the corresponding Poisson brackets of the coordinate functiosn j_a and p_a are

$$\{j_a, j_b\} = \epsilon_{abc} j_c, \qquad \{j_a, p_b\} = \epsilon_{abc} p_c. \tag{2.14}$$

In order to introduce m particles with masses and spins $(\mu_1, s_1), ...(\mu_m, s_m)$ we thus mark m points $z_1, ..., z_m$ on Σ and associate to each point z_i a co-adjoint orbit $\mathcal{O}_{\mu_i s_i}, i = 1, ...m$. The coupling of the particle degrees of freedom to the gauge field via minimal coupling is described in [5]. The upshot is that we specify the kinematic state of each particle by picking elements $\xi^*_{(i)} = p^a_{(i)} P^*_a + j^a_{(i)} J^*_a \in \mathcal{O}_{\mu_i s_i}$. The dual $iso(3)$ elements $\xi_{(i)} = p^a_{(i)} J_a + j^a_{(i)} P_a$ then act as the sources of curvature at each of the marked points:

$$F = \sum_{i=1}^{m} (p^a_{(i)} J_a + j^a_{(i)} P_a)\delta(z - z_i). \tag{2.15}$$

Expanding the curvature term as in (2.8) we find that the energy-momentum vectors of the particles act as sources for curvature and their generalised angular momenta act as sources of torsion, in agreement with physical expectations.

The above discussion can be generalised to include a non-vanishing cosmological constant $\lambda \in \mathbb{R}$ and Lorentzian gravity. We refer the reader to [9] and [15] for details. The idea is again to combine the spin connection and the dreibein into a Cartan connection. The form of the Cartan connection remains (2.7) but the Lie algebra structure of the space spanned by the generators J_a and P_a is modified as follows

$$[J_a, J_b] = \epsilon_{abc}J^c, \quad [J_a, P_b] = \epsilon_{abc}P^c, \quad [P_a, P_b] = \lambda\epsilon_{abc}J^c. \tag{2.16}$$

Here indices are raised with the Lorentzian metric $\eta^{ab} = (1, -1, -1)$ for Lorentzian signature but with the trivial metric δ_{ab} for Euclidean signature. To arrive at the Chern-Simons formulation we interpret the Cartan connection again as an Ehresmann connection on a bundle with a bigger structure group. The structure groups which result for the various values of λ and the two choices of signature are summarised in table 1.

Cosmological constant	Euclidean signature	Minkowskian signature
$\lambda = 0$	$ISO(3)$	$ISO(2,1)$
$\lambda > 0$	$SO(4) \simeq \frac{SU(2)\times SU(2)}{\mathbb{Z}_2}$	$SO(3,1) \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$
$\lambda < 0$	$SO(3,1) \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$	$SO(2,2) \simeq \frac{SL(2,\mathbb{R})\times SL(2,\mathbb{R})}{\mathbb{Z}_2}$

Table 1

Using the generalised commutation relations (2.16) and again the bilinear form (2.9) to expand the Chern-Simons action (2.10) one indeed finds the Einstein-Hilbert action for cosmological constant λ:

$$S_{EH}[\omega, e, \lambda] = \int_M e_a F^a_\omega + \lambda \int_M \epsilon_{abc}\, e^a \wedge e^b \wedge e^c. \tag{2.17}$$

3 Gravitational phase space and its symplectic structure

The phase space of a classical field theory is the space of solutions of the equations of motions, modulo gauge invariance. Adopting the Chern-Simons formulation of three dimensional gravity we thus find that the phase space of gravity in three dimensions is the moduli space of flat G-connection on the surface Σ, where G is the relevant gauge group extracted from table 1. In the following we specialise to compact Riemann surfaces of genus g. Also, we denote the Lie algebra of G by $\mathfrak{g}$. Starting with the classic paper of Atiyah and Bott [16], the moduli space of flat G-bundles on Σ has been studied extensively for semi-simple, compact Lie groups G. A pedagogical summary with further references can be found in [17].

The Lie groups relevant for our discussion of gravity are typically not compact and only some are semi-simple. This means in particular that the topology of various quotient spaces we are about to consider needs to be re-examined carefully. We shall restrict ourselves to a formal description of the relevant moduli spaces, but will highlight some points which call for a more careful analysis. That a formal discussion is possible at all depends crucially on the existence of the non-degenerate invariant form (2.9) on the Lie algebras relevant for us. Using that form, we briefly summarise the main results and formulae.

The tangent space to the space of G connections on a Riemann surface Σ is the affine space of $\mathfrak{g}$-valued one-forms on Σ. There is a natural symplectic form on this space, which can also be derived from the Chern-Simons action (2.10). It is

$$\Omega = \int_\Sigma \langle \delta_1 A \wedge \delta_2 A \rangle. \tag{3.1}$$

One checks that it is invariant under the gauge transformations

$$A \to gAg^{-1} + dgg^{-1}, \tag{3.2}$$

where $g \in$Map(Σ, G). An elementary calculation shows that the momentum mapping of this action is proportional to the curvature F. Hence the space of flat connections modulo gauge equivalence is equal to the symplectic quotient of the space of all G-connections by the group of gauge transformations (3.2). This quotient is the moduli space of flat G-connections on Σ. It inherits a symplectic structure from the symplectic structure on the space of all connections, which we refer to as the Atiyah-Bott symplectic structure. There are various ways of describing the moduli space and its symplectic structure. For us, the following description in terms of representations of the fundamental group is most useful. One of its advantages is that it is straightforward to include marked points on the Riemann surface. In the following we thus write $\Sigma_{g,m}$ for a compact Riemann surfact of genus g with m marked points.

Let π be the fundamental group of $\Sigma_{g,m}$. This group is generated by $2g+m$ invertible generators $a_1, b_1, ...a_g, b_g, l_1, ...l_m$ satisfying the relation

$$[b_g, a_g^{-1}]...[b_1, a_1^{-1}]l_m...l_1 = 1, \tag{3.3}$$

where $[x, y] = xyx^{-1}y^{-1}$. A flat G-connection on $\Sigma_{g,m}$ associates to each generator a holonomy element in G. However, the insertion of the charges at marked points as in (2.15) means that the holonomy around the i-th marked point is forced to lie in a fixed conjugacy class $\mathcal{C}_i$. We shall discuss in detail how the conjugacy classes are related to the co-adjoint orbits of the $\mathfrak{g}^*$-elements (2.12) in the next section.

The moduli space of flat connections on $\Sigma_{g,m}$ depends on these conjugacy classes, as well as on the group G and the genus g. We define it as

$$M(G, g, \mathcal{C}_1, ...\mathcal{C}_m) = \{\rho \in \mathrm{Hom}(\pi, G), \rho(l_i) \in \mathcal{C}_i\}/G, \tag{3.4}$$

where the group G acts via conjugation. In the case where G is compact and semi-simple this space is a Hausdorff space, and it is a manifold at irreducible points (where the image of π generates G). It would clearly be important to know if similar results hold for the groups G in table 1.

For our approach to the eventual quantisation of this space it is important that it can also be written as a quotient of the space $G^{2g} \times \mathcal{C}_1 \times ...\mathcal{C}_m$. To see this we introduce the group-valued momentum mapping $\mu : G^{2g} \times \mathcal{C}_1 \times ...\mathcal{C}_m \to G$,

$$\mu(A_1, B_1, ...A_g, B_g, L_1, ...L_m) = [B_g, A_g^{-1}]...[B_1, A_1^{-1}]L_m...L_1. \tag{3.5}$$

Then

$$M(G, g, \mathcal{C}_1, ...\mathcal{C}_m) = \mu^{-1}(1)/G. \tag{3.6}$$

The important observation of of Fock and Rosly [4] is that, under certain conditions, the space G^{2g+m} has a natural Poisson structure. When restricted to conjugation invariant functions on $G^{2g} \times \mathcal{C}_1 \times ...\mathcal{C}_m$ (which therefore descend to functions on $M(G, g, \mathcal{C}_1, ...\mathcal{C}_m)$) that Poisson structure agrees with the Poisson structure derived from the Atiyah-Bott symplectic structure on the moduli space $M(G, g, \mathcal{C}_1, ...\mathcal{C}_m)$. To write down the Fock-Rosly Poisson structure one requires an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ which satisfies the classical Yang-Baxter equation and is such that its symmetric part agrees with the non-degenerate invariant form $\langle \ , \ \rangle$ used in the definition of the symplectic structure (3.1). In the next section we shall write down such an r-matrix for $\mathfrak{g} = iso(3)$.

4 The Lie bi-algebra structure of $iso(3)$

In the following discussion we use the conventions and terminology of [18]. In particular we write $\sigma : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ for the flip operation $\sigma(X \otimes Y) = Y \otimes X$, and if $r = r^{\alpha\beta} X_\alpha \otimes X_\beta \in \mathfrak{g} \otimes \mathfrak{g}$, then $r_{12} \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ is defined to be $r_{12} = r^{\alpha\beta} X_\alpha \otimes X_\beta \otimes 1$.

Recall that a bi-algebra structure on a Lie algebra $\mathfrak{g}$ is a skew-symmetric linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, called the co-commutator, which satisfies the co-cycle condition

$$\delta[X, Y] = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)\delta(Y)) - (\mathrm{ad}_Y \otimes 1 + 1 \otimes \mathrm{ad}_Y)\delta(X)$$

and is such that the dual $\delta : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a commutator. A Lie bi-algebra $\mathfrak{g}$ is co-boundary if the co-commutator δ can be written as $\delta(X) = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)r$, for some element $r \in \mathfrak{g} \otimes \mathfrak{g}$ which satisfies two conditions. The first is that $r + \sigma(r)$ is an invariant element of $\mathfrak{g} \otimes \mathfrak{g}$. The second condition is that

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \tag{4.1}$$

is an invariant element of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ When that invariant element is 0, r is said to satisfy the classical Yang-Baxter equation (CYBE) and the bi-algebra is said to be quasi-triangular.

To exhibit the relevant Lie bi-algebra structure of $iso(3)$ we first recall the commutation relations for the rotation generators J_a and the translation generators P_a of $iso(3)$ given in (2.1) and (2.6). Further, we identify the vector space dual $iso(3)^*$ with $iso(3)$ via the non-degenerate pairing (2.9) on $iso(3)$. Note that the basis elements dual to the rotation generators are the translation generators and *vice-versa*.

The Lie algebra $\mathfrak{g} = iso(3)$ has several bi-algebra structures [19]. The one which is relevant for us is associated to the interpretation of $iso(3)$ as the classical double of the Lie algebra $so(3)$ (see [18] for a discussion of classical doubles). Note that this double structure is different from the bi-algebra structure one obtains by thinking of $iso(3)$ as a Wigner contraction of $so(4) = so(3) \oplus so(3)$ and using the standard bialgebra structure of $so(3) \simeq sl_2$. Quantising the latter leads to the q-deformed universal enevloping algebra $U_q(iso(3))$ as described for example in [20]. Quantising $iso(3)$ viewed as a classical double yields, as we shall see, a different quantum group, namely the quantum double $D(SU(2))$. For a classical double, there is a particularly simple formula for the classical r-matrix. In our case it is

$$r = P_a \otimes J_a. \tag{4.2}$$

One easily checks that it satisfies the CYBE and that the symmetrised part

$$r^s := \frac{1}{2}(r + \sigma(r)) = \frac{1}{2}(P_a \otimes J_a + J_a \otimes P_a) \tag{4.3}$$

is invariant under the adjoint action of $iso(3)$. Note in particular that r^s corresponds to the non-degenerate pairing (2.9) on the vector space $iso(3)$. We conclude that the bi-algebra $iso(3)$ is co-boundary and quasi-triangular, and that its co-commutator $\delta : iso(3) \to iso(3) \otimes iso(3)$ is

$$\delta(P_a) = \epsilon_{abc} P_b \otimes P_c, \qquad \delta(J_a) = 0. \tag{4.4}$$

Using the pairing (2.9) we find that the dual map $\delta^* : iso(3)^* \otimes iso(3)^* \to iso(3)^*$ is

$$\delta^*(J_a, J_b) = \epsilon_{abc} J_c \qquad \delta^*(J_a, P_b) = \delta^*(P_a, P_b) = 0. \tag{4.5}$$

This defines the Lie algebra structure of $iso(3)^*$. We deduce the Lie-algebra isomorphism $iso(3)^* \simeq so(3) \oplus \mathbb{R}^3$.

The classical r-matrix (4.2) is the only input needed to write down the Fock-Rosly Poisson structure on the phase space of three-dimensional Euclidean gravity. To review the Fock-Rosly structure briefly in a more general setting, let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. Fock and Rosly first gave a general formula for the Poisson bracket of functions on G^{2g+m} in [4]. A particularly convenient form can be found in [3]. It involves the vector fields generated by left- and right action of the group G on G^{2g+m}. The general expresssion is quite complicated and we will not write it down here. It is, however, instructive to write it down in the simplest case $g = 0$ and $m = 1$. In that case we obtain a Poisson structure on the group itself. Let $\{X_\alpha\}_{\alpha=1,\ldots \dim\mathfrak{g}}$ be a basis of $\mathfrak{g}$ and write X^L_α and X^R_α for the vector fields on G generated by left- and right action of $\mathfrak{g}$ on G. For a function f on G we define more precisely

$$X^L_\alpha f(g) = \frac{d}{dt} f(e^{-tX_\alpha} g)|_{t=0} \quad \text{and} \quad X^R_\alpha f(g) = \frac{d}{dt} f(g e^{tX_\alpha})|_{t=0}. \tag{4.6}$$

Then, if $r = r^{\alpha\beta} X_\alpha \otimes X_\beta$ is a solution of the CYBE, the Fock-Rosly bracket of two functions f_1 and f_2 on g is

$$\begin{aligned} \{f_1, f_2\} &= \frac{1}{2} r^{\alpha\beta}(X^R_\alpha f_1 X^R_\beta f_2 + X^L_\alpha f_1 X^L_\beta f_2 - X^R_\beta f_1 X^R_\alpha f_2 - X^L_\beta f_1 X^L_\alpha f_2) \\ &+ r^{\alpha\beta}(X^R_\alpha f_1 X^L_\beta f_2 - X^L_\beta f_1 X^R_\alpha f_2). \end{aligned} \tag{4.7}$$

To compute this bracket for the case $\mathfrak{g} = iso(3)$, we need some notation. The double cover $G = ISO(3)^{\sim}$ is the semi-direct product $\mathbb{R}^3 \rtimes SU(2)$. We write elements of $ISO(3)^{\sim}$ as pairs $(\boldsymbol{a}, u)$ of translation vectors $\boldsymbol{a} \in \mathbb{R}^3$ and elements $u \in SU(2)$. For an element $u \in SU(2)$ we write $\mathrm{Ad}(u)$ for the $SO(3)$ matrix representing u in the adjoint representation. Then the multiplication law for elements $(\boldsymbol{a}, u), (\boldsymbol{b}, v) \in ISO(3)^{\sim}$ is

$$(\boldsymbol{a}, u) \cdot (\boldsymbol{b}, v) = (\boldsymbol{a} + \mathrm{Ad}(u)\boldsymbol{b}, uv). \tag{4.8}$$

In the present context it is useful to parametrise group elemens via the exponential map. More precisely, we first identify an element $\xi^* \in iso(3)^*$ with an element $\xi \in iso(3)$ via the inner product (2.9) and then map it into $ISO(3)^{\sim}$ via the exponential map. The combined map is

$$\begin{aligned} \exp \circ * \ : \ & iso(3)^* \to ISO(3)^{\sim} \\ & p_a P_a^* + j_a J_a^* \mapsto \exp(j_a P_a + p_a J_a). \end{aligned} \tag{4.9}$$

To make contact with our previous parametrisation of $ISO(3)^{\sim}$ elements in terms of translations $\boldsymbol{a}$ and rotations u note that if $\exp(j_a P_a + p_a J_a)$ is the element $(\boldsymbol{a}, u)$, then $u = \exp(p_a J_a)$ and $\mathrm{Ad}(u^{-1})\boldsymbol{a} = \boldsymbol{j}$. Since the exponential map from $so(3)$ to $SU(2)$ is onto, it follows that the exponential map (4.9) is onto. Recalling our description of the co-adjoint orbits of $ISO(3)^{\sim}$ following eq. (2.13) we conclude that the orbits $\mathcal{O}_{\mu s}$ with $0 \leq \mu < 2\pi$ are mapped bijectively to conjugacy classes in $ISO(3) \sim$, which we denote by $\mathcal{C}_{\mu s}$. The only conjugacy classes in $ISO(3) \sim$ which are not bijectively related to co-adjoint orbits in this way are the classes containing elements with $u = -1$.

The vector fields J_a^L and P_a^L generated by left-action of, respectively, the rotations and translations on $ISO(3)^{\sim}$ are given by

$$\begin{aligned} J_a^L f(g) &= \frac{d}{dt} f(e^{-tJ_a} g)|_{t=0} \\ &= \left(\frac{\mathrm{Ad}(u)}{1 - \mathrm{Ad}(u)}\right)_{ab} \epsilon_{bcd} p_c \frac{\partial f}{\partial p_d}(g) \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} P_a^L f(g) &= \frac{d}{dt} f(e^{-tP_a} g)|_{t=0} \\ &= -\mathrm{Ad}(u)_{ab} \frac{\partial f}{\partial j_b}(g). \end{aligned} \tag{4.11}$$

In this and the following formulae, all functions are evaluated at $g = \exp(p_a J_a + j_a P_a)$. The vector fields generated by right actions of rotations and translations are

$$\begin{aligned} J_a^R f(g) &= \frac{d}{dt} f(g e^{tJ_a})|_{t=0} \\ &= \left(\frac{1}{\mathrm{Ad}(u) - 1}\right)_{ab} \epsilon_{bcd} p_c \frac{\partial f}{\partial p_d}(g) - \epsilon_{abc} j_b \frac{\partial f}{\partial j_c}(g) \end{aligned} \tag{4.12}$$

and

$$P_a^R f(g) = \frac{d}{dt} f(g e^{tP_a})|_{t=0} = \frac{\partial f}{\partial j_a}(g). \tag{4.13}$$

The Fock-Rosly Poisson bracket (4.7) of functions f_1 and f_2 on $ISO(3)^{\sim}$ now takes the following form.

$$\begin{aligned}\{f_1, f_2\} &= \frac{1}{2}\left(P_a^R f_1 J_a^R f_2 + P_a^L f_1 J_a^L f_2 - J_a^R f_1 P_a^R f_2 - J_a^L f_1 P_a^L f_2\right) \\ &\quad + P_a^R f_1 J_a^L f_2 - J_a^L f_1 P_a^R f_2 \\ &= \epsilon_{abc}\left(\frac{\partial f_1}{\partial p_a}\frac{\partial f_2}{\partial j_b}p_c + \frac{\partial f_1}{\partial j_a}\frac{\partial f_2}{\partial p_b}p_c\right) + \epsilon_{abc}\frac{\partial f_1}{\partial j_a}\frac{\partial f_1}{\partial j_b}j_c. \end{aligned} \tag{4.14}$$

In particular we find that the coordinate functions j_a and p_a, $a = 1, 2, 3$, have the Poisson brackets (2.14). We therefore conclude that the restriction of the Fock-Rosly Poisson bracket to a conjugacy classes $\mathcal{C}_{\mu s}$ in $ISO(3)^{\sim}$ with $\mu \notin 2\pi\mathbb{Z}$ is the push-forward of the Kostant-Kirillov Poisson structure on $\mathcal{O}_{\mu s}$ via the map (4.9).

5 The quantum double $D(SU(2))$ as a deformation of the group algebra of $ISO(3)^{\sim}$

The quantum double of a finite group H is a quasi-triangular Hopf algebra constructed, via Drinfeld's double construction, out of the Hopf algebra of functions on H. As a vector space $D(H)$ is the tensor product of the algebra of functions $F(H)$ on H and the group algebra $\mathbb{C}(H)$. The quantum double $D(H)$ plays an important role in the physics of orbifold conformal field theories [21] and in discrete gauge theories [22]. Its mathematical structure as a quasi-triangular Hopf algebra was first discussed in [23]. When generalising the construction to locally compact Lie groups H, one has to choose appropriate sets of (generalised) functions on H and different choices have been made by different authors. For a construction which emphasises the duality between $D(H)$ and $D(H)^*$ see [24] and for an approach via von Neumann algebras see [25]. In [26] a definition of $D(H)$ is given which emphasises the similarity with transformation group algebras. That approach is particularly well-adapted to the classification of the irreducible representations of $D(H)$, see [27]. For our purpose it is best to use a definition which is close to that of [26] but slightly more general. In this paper we are exclusively concerned with the case $H = SU(2)$. The main reference is [5], where the role of $D(SU(2)$ as the underlying symmetry of quantised $ISO(3)^{\sim}$ Chern-Simons theory was deduced from a detailed study of the quantum numbers, fusion and braiding properties of particles in that theory. However, the question of how to obtain the quantum group $D(SU(2))$ from a quantisation of the Poisson structure on the phase space of $ISO(3)^{\sim}$ Chern-Simons theory was not clarified. The goal of this section is to fill that gap. The chief difficulty which we have to overcome is the identification of the relevant deformation parameter. As we shall see presently, this is not manifest in the formulation of $D(SU(2))$.

We begin with a few general definitions. For a locally compact, unimodular group H with Haar measure dh we define $M(H)$ to be set of complex-valued, bounded measures on H which are absolutely continuous with respect to dh or pure point measures (thus we exclude continuous singular measures). For every absolutely continuous measure μ on H there is a function $f \in L^1(H, dh)$ such that $d\mu = f dh$. Pure point measures have no such representation, but for our calculations it is convenient to adopt the physicists' habit of writing pure point measures as a product of a Dirac delta function and the Haar measure. With that convention, the measures in $M(H)$ have the decomposition

$$d\mu = (f + \sum_{i \in \mathbb{N}} \lambda_i \delta_{h_i}) dh, \tag{5.1}$$

where $f \in L^1(H, dh)$, $\{\lambda_i\}_{i\in\mathbb{N}} \in l^1(\mathbb{C})$ and δ_{h_i} is the normlised Dirac delta function at h_i. $M(H)$ is a Banach algebra with norm

$$||\mu||_1 = \int_H |d\mu| \tag{5.2}$$

and multiplication defined by convolution. The convolution can be defined without reference to the decomposition (5.1) (see e.g. [28]), but using it simplifies the notation. Representing $d\mu_1$ and $d\mu_2$ by generalised functions f_1, and f_2 which are in $L^1(H)$, or delta functions or a linear combination of the two, the convolution becomes the ordinary convolution of functions:

$$f_1 * f_2(h) = \int_H dv f_1(v) f_2(v^{-1}h). \tag{5.3}$$

The algebra $M(H)$ will play the role of the group algebra of H in the rest of the paper. The role of the function algebra will be played by bounded, uniformly continuous functions $C_B(H)$ on H. When equipped with the supremum norm this, too, is a Banach algebra Then we define the quantum double $D(H)$ of a locally compact Lie group H with Haar measure dh as the set of bounded, $C_B(H)$-valued measures on H which are either absolutely continuous with respect to dh or pure point measures. More practically we think of elements of $D(H)$ as generalised functions F on $H \times H$ which are bounded and continuous in the first argument and measures of the form (5.1) in the second argument. The norm of such a function is

$$||F||_1 = \int_H dv \ \mathrm{sup}_{h\in H} |F(h, v)|. \tag{5.4}$$

We refer the reader to [26] and [5] for a complete list of how the algebraic operations of a quasi-triangular Hopf-$*$-algebra are implemented in $D(H)$. Note that the class of functions considered there is smaller than the ones included in our definition of $D(H)$. Some of the formula in [26] involve δ-functions and only make rigorous sense in our definition of $D(H)$. For our purposes we only need the formulae for the multiplication and the co-multiplication. The multiplication of two elements F_1 and F_2 of $D(H \times H)$ is

$$(F_1 \bullet F_2)(h, u) = \int_H dw \, F_1(h, w) F_2(w^{-1}hw, w^{-1}u). \tag{5.5}$$

The co-multiplication Δ is defined via

$$\Delta F(h_1, u_1, h_2, u_2) = F(h_1 h_2, u_1) \delta_{u_1}(u_2). \tag{5.6}$$

The final ingredient we need is the universal R-element:

$$R(h_1, u_1, h_2, u_2) = \delta_{h_1}(u_2) \delta_e(u_1). \tag{5.7}$$

The irreducible representations (irreps) of $D(SU(2))$ are classified in [26] and are labelled by pairs (μ, s), where $\mu \in [0, 2\pi]$ labels a conjugacy class C_μ in $SU(2)$ and s labels an irrep of the centraliser of a chosen point on the conjugacy class. Explicitly, we realise the rotation generators J_a as $J_a = -\frac{i}{2}\tau_a$, where the τ_a are the Pauli matrices. Then every element $h \in SU(2)$ can be written in axis-angle parametrisation as

$$h(\mu, \boldsymbol{n}) = \exp(\mu(n_a J_a)) \tag{5.8}$$

for a unit vector (n_1, n_2, n_3) and $\mu \in [0, 2\pi]$. There are two single-element conjugacy classes, $C_1 = \{1\}$ and $C_{-1} = \{-1\}$. In both cases the centraliser group is the whole of $SU(2)$, whose irreps are labelled by a positive half-integer, the spin. The other conjugacy classes are topological two-spheres, consisting of all rotations by some angle $\mu \in (0, 2\pi)$ about an arbitrary axis: $C_\mu = \{h(\mu, \boldsymbol{n}) | \boldsymbol{n} \in S^2\}$. In each class C_μ we single out the rotation about the 3-axis $h_\mu := \exp(\mu J_3)$ to define the centraliser group belonging to C_μ. That group therefore consists of all rotations around the 3-axis and is isomorphic to $U(1)$. Its representations are labelled by a half-integer, which one may think of as a "Euclidean helicity" in the present context. We denote them by π_s. For $\mu \in (0, 2\pi)$ and $s \in \mathbb{Z}/2$ let

$$H_s = \{\phi \in L^2(SU(2), \mathbb{C}) | \phi(x h_\omega) = \pi_s(h_\omega^{-1})\phi(x)\}. \tag{5.9}$$

This is the carrier space of the representation $\Pi_{\mu s}$. The action of an element $F \in D(SU(2))$ is

$$(\Pi_{\mu s}(F)\phi)(x) = \int_{SU(2)} dw\, F(x h_\mu x^{-1}, w)\phi(w^{-1}x). \tag{5.10}$$

The irreps labelled by $(0, j)$, $j = 0, 1/2, 1...$ are the standard spin j representations of $SU(2)$ with carrier space $\mathbb{C}^{2j+1}$. In order to simplify formulae in the following sections it is convenient to realise this carrier space as a certain space H_j of functions on $SU(2)$. Writing $D^j(x)$ for the matrix representing $x \in SU(2)$ in the standard spin j representation and $|j, j\rangle$ for the highest weight vector in that representation, we define a function ϕ for every $\varphi \in \mathbb{C}^{2j+1}$ via $\phi(x) = \langle j, j | D^j(x^{-1}) | \varphi \rangle$. H_j is the linear space of functions obtained from $\mathbb{C}^{2j+1}$ in this way. Equivalently one could define it as the span of the Wigner functions D^j_{nj}, $n = -j, -j+1, ..., j-1, j$ on $SU(2)$. Then the action of an element $F \in D(SU(2))$ on $\phi \in H_j$ is

$$(\Pi_j(F)\phi)(x) = \int_{SU(2)} dw\, F(1, w)\phi(w^{-1}x). \tag{5.11}$$

Finally we note the action of the universal R-element (5.7) in the tensor product representation $\Pi_{\mu_1 s_1} \otimes \Pi_{\mu_2 s_2}$ on some state $\Phi \in H_{s_1} \otimes H_{s_2}$. It is

$$(\Pi_{\mu_1, s_1} \otimes \Pi_{\mu_2, s_2}(R)\, \Phi)(x_1, x_2)) = \Phi(x_1, x_1 h_{\mu_1} x_1^{-1} x_2). \tag{5.12}$$

In order to establish the relationship between $D(SU(2))$and $ISO(3)^{\sim}$ we need to review some properties of $ISO(3)^{\sim}$. As a manifold $ISO(3)^{\sim} \simeq \mathbb{R}^3 \times SU(2)$, and the Haar measure on $ISO(3)^{\sim}$ is the product of the Lebesgue measure on $\mathbb{R}^3$ and the Haar measure on $SU(2)$. Concretely, if $g = (\boldsymbol{a}, u) \in ISO(3)^{\sim}$ then we use the Haar measure

$dg = d^3\boldsymbol{a}\, du$. Then we can realise the group algebra of $ISO(3)^\sim$ as described earlier for general locally compact groups as the set $M(ISO(3)^\sim)$ of bounded measures on $ISO(3)$ either absolutely continuous with respect to dg or pure point. Again representing such measures by generalised functions $\hat{f}_1, \hat{f}_2$ (possibly including delta-functions) the multiplication rule is the convolution

$$(\hat{f}_1 \hat{\bullet} \hat{f}_2)(\boldsymbol{a}, u) = \frac{1}{2\pi^3} \int_{\mathbb{R}^3 \times SU(2)} d^3b\, dw\, \hat{f}_1(\boldsymbol{b}, w) \hat{f}_2(\mathrm{Ad}(w^{-1})(\boldsymbol{a} - \boldsymbol{b}), w^{-1}u). \tag{5.13}$$

Now we perform a Fourier transform on the first argument of $\hat{f}$, thus obtaining a function f on $(\mathbb{R}^3)^* \times SU(2)$:

$$f(\boldsymbol{k}, x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3a\, \exp(-i\boldsymbol{k}\cdot\boldsymbol{a}) \hat{f}(\boldsymbol{a}, x). \tag{5.14}$$

Elements of the dual space $(\mathbb{R}^3)^*$ have the dimension of inverse length. Eventually they will physically be interpreted as momenta. Strictly speaking this is only possible after the introduction of a constant of the dimension length$\times$momentum, such as Planck's constant. Anticipating a more detailed discussion of dimensions and physical interpretations in the next section we refer to $(\mathbb{R}^3)^*$ as momentum space from now onwards.

The Fourier transform of a bounded measure on $\mathbb{R}^3$ is a bounded, uniformly continuous function on $(\mathbb{R}^3)^*$. Although not all bounded, uniformly continuous functions are obtained in this way, we define the group algebra of $ISO(3)^\sim$ as the set of of bounded measures on $SU(2)$, absolutely continuous with respect to the Haar measure or pure point, taking values in the Banach space of bounded, uniformly continuous functions on $(\mathbb{R}^3)^*$. As in the case of $D(SU(2))$ we think concretely of generalised functions f on $(\mathbb{R}^3)^* \times SU(2)$ which are bounded, uniformly continuous functions of the first argument and measures of the form (5.1) with respect to the second. For reasons which will become clear later we denote the set of all such generalised functions by A_0. We give A_0 the structure of a Hopf algebra as follows. The product of two generalised functions f_1, f_2 in A_0 is obtained by applying the Fourier transform (5.14) to the convolution product. The result is

$$(f_1 \bullet f_2)(\boldsymbol{k}, u) = \int_{SU(2)} dw\, f_1(\boldsymbol{k}, w) f_2(\mathrm{Ad}(w^{-1})\boldsymbol{k}, w^{-1}u). \tag{5.15}$$

The group-like co-multiplication for $ISO(3)^\sim$ leads to the following co-multiplication for $f \in A_0$

$$(\Delta f)(\boldsymbol{k}_1, u_1, \boldsymbol{k}_2, u_2) = f(\boldsymbol{k}_1 + \boldsymbol{k}_2, u_1)\delta_{u_1}(u_2). \tag{5.16}$$

One checks that with antipode

$$Sf(\boldsymbol{k}, u) = f(-\mathrm{Ad}(u^{-1})\boldsymbol{k}, u^{-1}), \tag{5.17}$$

unit

$$1(\boldsymbol{k}, u) = \delta_e(u) \tag{5.18}$$

and co-unit

$$\epsilon(f) = \int_{SU(2)} f(0,u) \tag{5.19}$$

A_0 is a co-commutative Hopf algebra.

The irreps of the Euclidean group $ISO(3)^{\sim}$ and hence also of A_0 are labelled by pairs of $SU(2)$ orbits and centraliser representations, much like for the double $D(SU(2))$. Now the relevant orbits are the orbits of the $SU(2)$ action on $(\mathbb{R}^3)^*$. There is one orbit consisting of the origin $(0,0,0)$, and all the other orbits are two-spheres in $(\mathbb{R}^3)^*$. These orbits are therefore naturally labelled by their radius. In the current context this radius has the interpretation of Euclidean mass, so we denote it by $M \in \mathbb{R}^{\geq 0}$. If $M = 0$ the centraliser group is the whole of $SU(2)$, with irreps labelled by the spin j as before. For $M > 0$ we single out the point $\boldsymbol{d}_3^M = (0,0,M)$ on the 3-axis. Then the centraliser group is the group of rotations around the 3-axis. The generic irreps of $ISO(3)^{\sim}$ are therefore labelled by pairs (M,s), with $M > 0$ and $s \in \mathbb{Z}/2$ labelling an irrep of $U(1)$. The carrier spaces of such irreps are the Hilbert spaces H_s defined in (5.9). The action of $f \in A_0$ is

$$(\Pi_{Ms}(f)\phi)(x) = \int_{SU(2)} dw\, f(\boldsymbol{k}(x,M),w)\phi(w^{-1}x), \tag{5.20}$$

where

$$\boldsymbol{k}(x,M) = \mathrm{Ad}(x)\boldsymbol{d}_3^M. \tag{5.21}$$

Note that elements $(\boldsymbol{a},u) \in ISO(3)^{\sim}$ correspond to plane waves in momentum space and δ-functions in $SU(2)$: $f_{\boldsymbol{a},u}(\boldsymbol{k},v)) = \exp(i\boldsymbol{a}\cdot\boldsymbol{k})\,\delta_u(v)$. Applying the formula (5.20) to such functions one obtains the familiar formula for the representation of $(\boldsymbol{a},u) \in ISO(3)^{\sim}$

$$(\Pi_{Ms}((\boldsymbol{a},u))\phi)(x) = \exp(i\boldsymbol{a}\cdot\boldsymbol{k}(x,M))\phi(u^{-1}x). \tag{5.22}$$

From the representation of the Lie group, we obtain representations of the Lie algebra in the usual way. Writing $\boldsymbol{d}_a$, $a = 1,2,3$, for the canonical basis of $(\mathbb{R}^3)^*$ ($\boldsymbol{d}_1 = (1,0,0)$ etc.) we have the following representation of generators P_a of translations:

$$(\Pi_{Ms}(P_a)\phi)(x) = \frac{d}{d\epsilon}(\Pi_{Ms}((\epsilon\boldsymbol{d}_a,1))\phi)(x)|_{\epsilon=0} = i\,k_a(x,M)\phi(x). \tag{5.23}$$

For the rotation generators J_a we find

$$(\Pi_{Ms}(J_a)\phi)(x) = \frac{d}{d\epsilon}(\Pi_{Ms}((0,\exp(\epsilon J_a)))\phi)(x)|_{\epsilon=0}, = -\mathcal{J}_a\phi(x) \tag{5.24}$$

where $\mathcal{J}_a$, $a = 1,2,3$, are the vector fields on $SU(2)$ which generate the left action of $SU(2)$ on itself. We deduce in particular that the classical r-matrix (4.2) acts on elements Φ of the tensor product $H_{s_1} \otimes H_{s_2}$ via

$$((\Pi_{M_1s_1} \otimes \Pi_{M_2s_2})(r)\Phi)(x_1,x_2) = -i(k_a(x_1,M_1)\mathcal{J}_a^{(2)})\Phi(x_1,x_2), \tag{5.25}$$

where we use the superscript $^{(2)}$ to indicate that $\mathcal{J}^{(2)}$ acts on the second argument of Φ.

The final ingredient we need in order establish the promised relationship between $D(SU(2))$ and $ISO(3)^{\sim}$ is a family of exponential maps from the Lie algebra $so(3)$ to $SU(2)$. Let $J^\kappa_a = \kappa J_a$ for a real, positive parameter κ, so that

$$[J^\kappa_a, J^\kappa_b] = \kappa\epsilon_{abc} J^\kappa_a. \tag{5.26}$$

Then define

$$\exp_\kappa : (\mathbb{R}^3)^* \to SU(2), \quad \exp_\kappa(\boldsymbol{k}) = \exp(k_a J^\kappa_a). \tag{5.27}$$

This map is injective when restricted to the open Ball $B^3_{(2\pi/\kappa)}$ of radius $2\pi/\kappa$ in $(\mathbb{R}^3)^*$, but it is not surjective: the image of $B^3_{(2\pi/\kappa)}$ is $SU(2)\backslash\{-1\}$. Now let $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$ for $\boldsymbol{k} \neq 0$, and write $S^2_{(2\pi/\kappa)}$ for the two-sphere of radius $2\pi/\kappa$ in $(\mathbb{R}^3)^*$. Then define the space

$$V_\kappa = \{f \in C_B((\mathbb{R}^3)^*) | f|_{S^2_{2\pi/\kappa}} = \text{const},\; f(\boldsymbol{k} - (4\pi/\kappa)\hat{\boldsymbol{k}}) = f(\boldsymbol{k})\, \forall \boldsymbol{k} \in \mathbb{R}^3\backslash\{0\}\}. \tag{5.28}$$

Note that any $f \in V_\kappa$ necessarily takes the same value on all two-spheres of radius $4m\pi/\kappa$, $m \in \mathbb{Z}$ as at the origin. For us V_κ is important because it contains the image of the pull-back map

$$\exp^*_\kappa : C_B(SU(2)) \to C_B((\mathbb{R}^3)^*). \tag{5.29}$$

This map is injective and its inverse can be defined if we restrict it to V_κ. We denote the inverse by

$$\log^*_\kappa : V_\kappa \to C_B(SU(2)). \tag{5.30}$$

Finally we define, for fixed κ, the space A_κ as the space of V_κ-valued, bounded, absolutely continuous or pure point measures on $SU(2)$. Again we think of elements of A_κ concretely as functions on $(\mathbb{R}^3)^* \times SU(2)$ which are in V_κ as function of the first argument and of the form (5.1) in the second argument. It is clear that for each positive value of κ, $A_\kappa \subset A_0$ and that A_κ inherits an algebra structure from A_0. In fact A_κ is a consistent truncation of A_0 as an algebra: A_κ is closed under the multiplication $\bullet$ (5.15) and the unit (5.18) of A_0 is contained in A_κ. However, the co-multiplication Δ (5.16) does not map A_κ into $A_\kappa \otimes A_\kappa$, so that A_κ is not a consistent truncation of A_0 as a Hopf algebra. We shall now show how to turn A_κ into a quasi-triangular Hopf algebra by introducing a deformed co-multiplication.

To construct the new co-multiplication we extend the map (5.29) to a map

$$\mathrm{EXP}^*_\kappa : D(SU(2)) \to A_\kappa, \tag{5.31}$$

by pull-back on the first argument. This map is again invertible, and we denote the inverse by

$$\mathrm{LOG}^*_\kappa : A_\kappa \to D(SU(2)). \tag{5.32}$$

Now we can use the κ-dependent bijection EXP_κ to induce a Hopf-algebra structure from $D(SU(2))$ on A_κ. This re-produces the κ-independent multiplication rule (5.15), the antipode (5.17), unit (5.18) and co-unit (5.19). The co-multiplication $A_\kappa \to A_\kappa \otimes A_\kappa$, however, does depend on κ:

$$(\Delta_\kappa f)(\boldsymbol{k}_1, u_1, \boldsymbol{k}_2, u_2) = \mathrm{LOG}^*_\kappa(f(\exp_\kappa(\boldsymbol{k}_1)\exp_\kappa(\boldsymbol{k}_2), u_1))\ \delta_e(u_1^{-1}u_2). \tag{5.33}$$

For a better understanding of this formula and of the limit we will consider shortly, it is useful to think of it in terms of the Baker-Campell-Hausdorff expression for the product of exponentials of Lie algebra elements $X, Y \in su(2)$: $\exp(X)\exp(Y) = \exp(X * Y)$, where

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[[X,Y],Y] + ... \tag{5.34}$$

With $X = k_1^a J_a^\kappa$,$Y = k_2^a J_a^\kappa$, this induces a κ-dependent product on $(\mathbb{R}^3)^*$:

$$\boldsymbol{k}_1 * \boldsymbol{k}_2 = \boldsymbol{k}_1 + \boldsymbol{k}_2 + \frac{\kappa}{2}\boldsymbol{k}_1 \wedge \boldsymbol{k}_2 + \frac{\kappa^2}{12}(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2(\boldsymbol{k}_1 + \boldsymbol{k}_2) - \boldsymbol{k}_1^2\boldsymbol{k}_2 - \boldsymbol{k}_2^2\boldsymbol{k}_1) + ... \tag{5.35}$$

This is the product used in the first argument of $f \in A_\kappa$ to define the co-product in (5.33). Thinking of the $\boldsymbol{k}_1, \boldsymbol{k}_2$ as momenta we should think of (5.35) as a "curved" momentum addition, which reduces to the usual linear addition in the limit $\kappa \to 0$.

The universal R-element (5.7) of $D(SU(2)) \otimes D(SU(2))$ can be mapped into $A_\kappa \otimes A_\kappa$ via the pull-back $\mathrm{EXP}^*_\kappa \otimes \mathrm{EXP}^*_\kappa$. We denote the pull-back by R_κ, i.e.

$$R_\kappa = (\mathrm{EXP}^*_\kappa \otimes \mathrm{EXP}^*_\kappa)(R). \tag{5.36}$$

Thus we obtain a family $(A_\kappa, \bullet, \Delta_\kappa, 1, \epsilon, S, R_\kappa)$ of quasi-triangular Hopf algebras which are all isomorphic to $D(SU(2))$ for $\kappa > 0$. The last remark in the previous paragraph shows that the co-multiplication in A_κ tends to the co-commutative co-product of A_0 (5.16) in the limit $\kappa \to 0$.

The relation between the universal R-element R_κ and the r-matrix (4.2) of $iso(3)$ can now be established as follows. Since $R_\kappa \in A_\kappa \otimes A_\kappa \subset A_0 \otimes A_0$ we can let it act in an irrep $\Pi_{M_1,s_1} \otimes \Pi_{M_2,s_2}$ of $A_0 \otimes A_0$ and consider the limit $\kappa \to 0$. To keep the following formulae simple we write $\Pi_1 \otimes \Pi_2$ instead of $\Pi_{M_1 s_1} \otimes \Pi_{M_2 s_2}$. Then

$$R_\kappa(\boldsymbol{k}_1, y_1, \boldsymbol{k}_2, y_2) = \delta_e(\exp_\kappa(\boldsymbol{k}_1)y_2^{-1})\delta_e(y_1). \tag{5.37}$$

Acting on $\Phi \in H_{s_1} \otimes H_{s_2}$ via $\Pi_1 \otimes \Pi_2$ gives

$$(\Pi_1 \otimes \Pi_2(R_\kappa)\Phi)(x_1, x_2) = \Phi(x_1, \exp_\kappa(\boldsymbol{k}(x_1, M_1))\, x_2), \tag{5.38}$$

where we have used the notation (5.21). Then we find for small κ

$$(\Pi_1 \otimes \Pi_2(R_\kappa)\,\Phi)(x_1, x_2) = (1 \otimes 1 + \kappa\, k^a(x_1, M_1)\mathcal{J}_a^{(2)})\,\Phi(x_1, x_2)\ +\ \mathcal{O}(\kappa^2). \tag{5.39}$$

The term of order κ is the action of r in the representation $\Pi_1 \otimes \Pi_2$ (5.25). Since the above statement is true in any representation, we conclude that in the limit $\kappa \to 0$

$$R_\kappa = 1 \otimes 1 + i\kappa\, r +\ \mathcal{O}(\kappa^2). \tag{5.40}$$

6 Discussion and outlook

In the previous section we have implemented the key step of the combinatorial quantisation procedure for the case of Euclidean three dimensional gravity. The quantum R-matrix R_κ can now be used to define the graph algebra as introduced and discussed in [1] and [2]. The relations defining this algebra are numerous and complicated and will not be repeated here. From the graph algebra one obtains another algebra, introduced in [2] and called the moduli algebra $\mathcal{M}$ in [3], by taking a quotient and imposing various conditions. The moduli algebra is the quantisation of the commutative algebra of functions on the classical moduli space $M(G, g, \mathcal{C}_1, ..., \mathcal{C}_m)$, which in our case is interpreted as the gravitational phase space. The algbra $\mathcal{M}$ may thus be thought of as the algebra of observables in quantised three-dimensional gravity. It would clearly be very interesting to study this algebra in more detail, and to interpret it physically. This should probably be done in the case of Lorentzian gravity, where the physical interpretation is clearer. One important aspect of that discussion will be a clear understanding of the classical limit. Here we only note that this limit is related to the limit $\kappa \to 0$ of our deformation parameter, as can be seen by a simple dimensional analysis. In units where the speed of light c is 1, the gravitational coupling constant G_3 in three dimensions (Newton's constant) has the dimension of inverse mass. The elements $\boldsymbol{k}$ of the space $(\mathbb{R}^3)^*$ defined in the previous section have the dimension of inverse length. Since the argument of the exponential function (5.27) must be dimensionless, it follows that the deformation parameter κ has the dimension of length. Since G_3 and Planck's constant $\hbar$ are the only available physical constants in pure three dimensional quantum gravity (having set $c = 1$), it follows that κ is proportional to $\hbar G_3$. The classical limit $\hbar \to 0$ therefore corresponds, for fixed G_3, to the limit $\kappa \to 0$.

The final step in the quantisation program, carried out for compact gauge groups in [3], is the construction of a representation of the moduli algebra $\mathcal{M}$ on a suitable Hilbert space. The implementation of this step for three dimensional quantum gravity would again be of great interest. For Riemann surfaces with genus zero one almost certainly reproduces the results of [5], where the Hilbert space of m distinguishable particles on a genus zero surface is constructed from irreps of $D(SU(2))$. As pointed out in [5] the R-element of $D(SU(2))$ provides a representation of the braid group action on the multiparticle Hilbert space. As further explained in [31], the analogous braid group action in the Lorentzian case allows one to compute the gravitational quantum scattering of particles, at least in principle. In practice there are a number of conceptual and practical problems, addressed in [6]. Within the combinatorial quantisation programme it is not difficult also to consider particles on Riemann surfaces of higher genus. As explained in [3] one is guaranteed to obtain a Hilbert space which carries a representation of the mapping class group of the surface. In gravity, the requirement of diffeomorphism invariance implies that only states which are invariant under the action of the mapping class group are phyiscal. The scheme outlined here thus leads to a natural implementation of diffeomorphism invariance in quantum gravity, something which has proved difficult in other approaches.

Mathematically, a number of issues need to be resolved. The classical moduli spaces $M(G, g, \mathcal{C}_1, ..., \mathcal{C}_m)$ for the non-compact gauge groups in table 1 should be investigated rigorously, and both their symplectic and their singularity structure should be clarified. Here the more geometric approach to the symplectic structure developed in [29] and [30] may be useful. Looking ahead, a number of generalisations of our sects. 4 and 5 should be studied. The deformation of the group algebra of $ISO(3)^{\sim}$ to the quantum double $D(SU(2))$ described in sect. 5 of this paper can be generalised to semi-direct product groups of the form $\mathfrak{h} \rtimes H$, where H is an arbitrary locally compact unimodular Lie group and $\mathfrak{h}$ the Lie algebra of H, thought of as an abelian group under addition, with H acting via conjugation. Applying the procedure described in this paper, the group algebra of $\mathfrak{h} \rtimes H$ can be deformed to $D(H)$. More ambitiously, it would be interesting to include the cosmological constant in our discussion. In [7] the quantum group $SL_q(2,\mathbb{C})_{\mathbb{R}}$ was argued to be relevant in the combinatorial quantistation of $SL(2,\mathbb{C})$ Chern-Simons theory. This would suggest an interesting connection between the quantum group $SL_q(2,\mathbb{C})_{\mathbb{R}}$ and the quantum doubles $D(SU(2))$ and $D(SO(2,1))$.

Acknowledgments
The question addressed in this article arose in discussions I had with Sander Bais and Nathalie Muller while I was a post doc in the Institute for Theoretical Physics of the University of Amsterdam. Since then I have had further illuminating discussions with Anton Alekseev, Sander Bais, David Calderbank, Tom Koornwinder, Volker Schomerus, Allan Sinclair and Joost Slingerland. I thank the organisers of the workshop "Deformation quantisation of singular reduced spaces" in Oberwolfach for the opportunity to present an early version of this work, and acknowledge financial support through an Advanced Research Fellowship of the Engineering and Physical Sciences Research Council.

References

[1] A. Y. Alekseev, H. Grosse and V. Schomerus, *Combinatorial quantization of the Hamiltonian Chern-Simons Theory,* Commun. Math. Phys., **172** (1995), 317–358.

[2] A. Yu. Alekseev, H. Grosse and V. Schomerus, *Combinatorial quantization of the Hamiltonian Chern-Simons Theory II,* Commun. Math. Phys., **174** (1995), 561–604.

[3] A. Yu. Alekseev and V. Schomerus, *Representation theory of Chern-Simons observables,* Duke Math. Journal, **85** (1996), 447–510.

[4] V. V. Fock and A. A. Rosly, *Poisson structures on moduli of flat connections on Riemann surfaces and r-matrices,* ITEP preprint **72-92** (1992); see also `math.QA/9802054`.

[5] F. A. Bais and N. M. Muller, *Topological field theory and the quantum double of $SU(2)$,* Nucl. Phys., **B530** (1998), 349–400.

[6] F. A. Bais, N. M. Muller and B. J. Schroers, *Quantum double symmetry and topological interactions in (2+1)-dimensional quantum gravity*, in preparation.

[7] E. Buffenoir and Ph. Roche, *Harmonic analysis on the quantum Lorentz group,* Commun. Math. Phys., **207** (1999), 499–555.

[8] A. Achucarro and P. Townsend, *A Chern–Simons action for three-dimensional anti-de Sitter supergravity theories,* Phys. Lett., **B180** (1986), 85–100.

[9] E. Witten, *2+1 dimensional gravity as an exactly soluble system,* Nucl. Phys., **B311** (1988), 46–78.

[10] S. Carlip, *Quantum gravity in (2+1) dimensions,* Cambridge University Press, Cambridge, **1998**.

[11] S. Carlip, *Lectures on 2+1 dimensional gravity,* UCD-95-**6** (1995), `gr-qc/9503024`.

[12] R. W. Sharpe, *Differential Geometry,* Springer Verlag, New York, **1996**.

[13] H.-J. Matschull, *On the relation between (2+1) Einstein gravity and Chern-Simons Theory,* Class. Quant. Grav., **16** (1999), 2599–2609.

[14] J. E. Marsden and T. S. Ratiu, *Introduction to mechanics and symmetry,* Springer Verlag, New York, **1994**.

[15] E. Witten, *Quantization of Chern-Simons gauge theory with complex gauge group,* Commun. Math. Phys., **137** (1991), 29–66.

[16] M. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces,* Phil. Trans. Roy. Soc. London Ser. A, **308** (1983), 523–615.

[17] M. Atiyah, *The geometry and physics of knots,* Cambridge University Press, Cambridge, **1990**.

[18] V. Chari and A. Pressley *Quantum Groups,* Cambridge University Press, Cambridge, **1994**.

[19] P. Stachura, *Poisson-Lie structures on Poincaré and Euclidean groups in three dimensions,* J. J. Phys., **A 31** (1998), 4555–4564.

[20] E. Celeghini and R. Giachetti, *The three dimensional euclidean quantum group $E(3)_q$ and its R-matrix,* J. Math. Phys., **32** (1991), 1159–1165.

[21] R. Dijkgraaf, C. Vafa, E. Verlinde, and H. Verlinde, *The operator algebra of orbifold models,* Commun. Math. Phys., **123** (1989), 485–526.

[22] F. A. Bais, P. van Driel, and M. de Wild Propitius, *Quantum symmetries in discrete gauge theories,* Phys. Lett., **B 280** (1992), 63–70.

[23] R. Dijkgraaf, V. Pasquier, and P. Roche, *Quasi Hopf algebras, group cohomology and orbifold models,* Nucl. Phys. (Proc. Suppl.), **18B** (1990), 60–72.

[24] P. Bonneau, *Topological quantum double,* Rev. Math. Phys., **6** (1994), 305–318.

[25] M. Müger, *Quantum Double actions on operator algebras and orbifold quantum field theories,* Commun. Math. Phys., **191** (1998), 137–181.

[26] T. H. Koornwinder and N. M. Muller. *The quantum double of a (locally) compact group,* J. Lie Theory **7** (1997), 33–52;**8** (1998), 187 (erratum).

[27] T. H. Koornwinder, F. A. Bais and N. M. Muller, *Tensor Product Representations of the Quantum Double of a Compact Group,* Commun. Math. Phys., **198** (1998), 157–186.

[28] G. K. Pedersen, *C^*algebras and their automorphism groups,* Academic Press, London, **1979**.

[29] K. Guruprasad, J. Huebschmann, L. Jeffrey, A. Weinstein, *Group systems, groupoids, and moduli spaces of parabolic bundles,* Duke Math. J., **89** (1997), 377–412.

[30] A. Yu. Alekseev, A. Z. Malkin and E. Meinrenken, *Lie group valued moment maps,* J. Differential Geom., **48** (1998), 445–495.

[31] N. Muller, *Topological interactions and quantum double symmetries,* Ph.D. dissertation, University of Amsterdam, **1998**.