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Bogomol'nyi solitons in a gauged $O(3)$ sigma model

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Abstract

The scale invariance of the $O(3)$ sigma model can be broken by gauging a $U(1)$ subgroup of the $O(3)$ symmetry and including a Maxwell term for the gauge field in the Lagrangian. Adding also a suitable potential one obtains a field theory of Bogomol'nyi type with topological solitons. These solitons are stable against rescaling and carry magnetic flux which can take arbitrary values in some finite interval. The soliton mass is independent of the flux, but the soliton size depends on it. However, dynamically changing the flux requires infinite energy, so the flux, and hence the soliton size, remains constant during time evolution.

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1 Introduction

The $O(3)$ sigma model in (2+1) dimensions is a popular model in theoretical physics. Statically it is integrable and of Bogomol'nyi type, i.e. all minimal energy solutions can be obtained by solving the first order Bogomol'nyi equations (which imply the second order Euler-Lagrange equations). As a result one can explicitly write down soliton solutions of arbitrary degree in terms of rational functions [1]. From the point of view of a particle physicist, however, the model has one important drawback: it is scale invariant and as a result its soliton solutions have arbitrary size, making them unsuitable as models for particles. Numerical simulations of the solitons' interaction behaviour in the (2+1)-dimensional model suggest that the solitons do indeed change their size during interactions and generically turn into singular configurations of zero size [2]. The most obvious way to break the scale invariance of the model is to add terms to the Lagrangian density which contain a different number of derivatives from the sigma model term (which contains two). Indeed, the inclusion of a Skyrme term (four derivatives) and a potential term (no derivatives) leads to so-called baby Skyrme models which have soliton solutions of definite size. Such models are close analogues of the (3+1) dimensional Skyrme model and therefore physically interesting, but they are neither integrable nor of Bogomol'nyi type and can only be studied with considerable numerical effort [3]. A mathematically more elegant way of breaking the scale invariance of the $O(3)$ sigma model is to add a potential term only and prevent the solitons from collapsing by making them spin. In [4] it was shown that with a suitable choice of the potential such a model is of Bogomol'nyi type. Its soliton solutions, called Q-lumps, can be written down explicitly and their interaction behaviour was studied in [5].

Here we investigate the possibility of breaking the scale invariance of the sigma model by introducing a $U(1)$ gauge field whose dynamics is governed by a Maxwell term. This possibility is also potentially of interest in the (3+1)-dimensional Skyrme model. There the gauging of a $U(1)$ subgroup and the inclusion of a Maxwell term is physically natural, and it would be aesthetically appealing if one could do away with the Skyrme term and still retain a model with stable soliton solutions. Since in three spatial dimensions the sigma model term has the opposite scaling behaviour from the Maxwell term, simple scaling arguments do not rule out solitons of a definite size in a "Skyrme-Maxwell model without a Skyrme term". At the end of the paper we will make some conjectures about this possibility.

Since the solitons here carry magnetic flux it is appropriate to compare them to vortices. Vortices either have quantised flux in which case they are topologically stable (e.g. in the abelian Higgs model, see [6] and references therein) or they have arbitrary flux in which case they are not topologically stable (e.g. non-topological Chern-Simons vortices, see [7]). The topological stability of the solitons studied here, however, is independent of their magnetic flux. Thus they can have arbitrary flux and yet be topologically stable.

2 Topological stability

In the $O(3)$ sigma model the basic field is a map from (2+1)-dimensional Minkowski space to the 2-sphere of unit radius. Here Minkowski space is assumed to have the signature $(-,+,+)$, and its elements are written as $(t,\boldsymbol{x})$ or alternatively x^α, $\alpha=0,1,2$; for partial derivatives with respect to these coordinates we write ∂_α. The field, denoted $\boldsymbol{\phi}$, has three components ϕ_1,ϕ_2 and ϕ_3 satisfying the constraint $\boldsymbol{\phi}\cdot\boldsymbol{\phi}=\phi_1^2+\phi_2^2+\phi_3^2=1$. The potential energy functional is

$$E_\sigma[\boldsymbol{\phi}]=\frac{1}{2}\int d^2x\,\left((\partial_1\boldsymbol{\phi})^2+(\partial_2\boldsymbol{\phi})^2\right). \tag{2.1}$$

To obtain finite-energy configurations one requires that, at all times t,

$$\lim_{|\boldsymbol{x}|\to\infty}\boldsymbol{\phi}(t,\boldsymbol{x})=\boldsymbol{n}, \tag{2.2}$$

where $\boldsymbol{n}$ is a constant unit vector which we take to be $\boldsymbol{n}=(0,0,1)$ for definiteness. This condition allows one to add the point ∞ to physical space $\mathbf{R}^2$, thus compactifying it to a topological 2-sphere. As a result a field $\boldsymbol{\phi}$ at a fixed time may be viewed as a map from one 2-sphere to another and therefore has an associated degree $\deg[\boldsymbol{\phi}]$. This degree is a homotopy invariant and therefore cannot change during time evolution.

For our purposes it is convenient to express the degree in terms of the current

$$k_\alpha=\frac{1}{2}\epsilon_{\alpha\beta\gamma}\boldsymbol{\phi}\cdot\partial^\beta\boldsymbol{\phi}\times\partial^\gamma\boldsymbol{\phi}, \tag{2.3}$$

where $\epsilon_{\alpha\beta\gamma}$ is the totally antisymmetric tensor in three indices, normalised so that $\epsilon_{012}=1$. One finds

$$\deg[\boldsymbol{\phi}]=\frac{1}{4\pi}\int d^2x\,k_0=\frac{1}{4\pi}\int d^2x\,\boldsymbol{\phi}\cdot\partial_1\boldsymbol{\phi}\times\partial_2\boldsymbol{\phi}. \tag{2.4}$$

It is easy to see that the divergence of k_α vanishes independently of the equations of motion; together with the boundary condition (2.2) this explicitly shows the conservation of the degree. The degree of a configuration is also important because it provides a lower bound on its energy [1]:

$$E_\sigma[\boldsymbol{\phi}]\geq 4\pi\,|\deg[\boldsymbol{\phi}]|\,. \tag{2.5}$$

The energy functional E_σ and the boundary condition (2.2) are invariant under the group of global rotations of the field $\boldsymbol{\phi}$ about the fixed vector $\boldsymbol{n}$. This is the $U(1)$ symmetry we want to gauge. Thus we introduce a $U(1)$ gauge field A_α and a covariant derivative

$$D_\alpha\boldsymbol{\phi}=\partial_\alpha\boldsymbol{\phi}+A_\alpha\boldsymbol{n}\times\boldsymbol{\phi}. \tag{2.6}$$

Defining the field strength as usual via $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ with magnetic field F_{12}, we can write down the potential energy functional which is the subject of this paper:

$$E_{\text{gauge}}[\boldsymbol{\phi}, A_1, A_2] = \frac{1}{2}\int d^2x \, \Big((D_1\boldsymbol{\phi})^2 + (D_2\boldsymbol{\phi})^2 + (1 - \mathbf{n}\cdot\boldsymbol{\phi})^2 + F_{12}^2\Big) \,. \tag{2.7}$$

In the gauged model the topological current k_α, while still divergence free, is unsatisfactory because it is not gauge invariant. Through trial and error one finds that the current

$$j_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma}\left(\boldsymbol{\phi}\cdot D^\beta\boldsymbol{\phi} \times D^\gamma\boldsymbol{\phi} + F^{\beta\gamma}(1 - \mathbf{n}\cdot\boldsymbol{\phi})\right), \tag{2.8}$$

which is manifestly gauge invariant, differs from k_α only by the curl of another vector field

$$j_\alpha = k_\alpha + \epsilon_{\alpha\beta\gamma}\partial^\beta\left((1 - \mathbf{n}\cdot\boldsymbol{\phi})A^\gamma\right). \tag{2.9}$$

Thus j_α also has vanishing divergence, $\partial_\alpha j^\alpha = 0$, and together with the boundary condition (2.2) this implies that the degree of a configuration $\boldsymbol{\phi}$ can be expressed as

$$\deg[\boldsymbol{\phi}] = \frac{1}{4\pi}\int d^2x \, j_0 = \frac{1}{4\pi}\int d^2x \, \left(\boldsymbol{\phi}\cdot D_1\boldsymbol{\phi} \times D_2\boldsymbol{\phi} + F_{12}(1 - \mathbf{n}\cdot\boldsymbol{\phi})\right). \tag{2.10}$$

Returning to the energy functional E_{gauge} we note that it can be re-written as

$$\begin{aligned} E_{\text{gauge}}[\boldsymbol{\phi}, A_1, A_2] = {} & \frac{1}{2}\int d^2x \, \Big((D_1\boldsymbol{\phi} \pm \boldsymbol{\phi} \times D_2\boldsymbol{\phi})^2 + (F_{12} \mp (1 - \mathbf{n}\cdot\boldsymbol{\phi}))^2\Big) \\ & \pm \int d^2x \, \left(\boldsymbol{\phi}\cdot D_1\boldsymbol{\phi} \times D_2\boldsymbol{\phi} + F_{12}(1 - \mathbf{n}\cdot\boldsymbol{\phi})\right), \end{aligned} \tag{2.11}$$

where we used that $\boldsymbol{\phi}\cdot D_\alpha\boldsymbol{\phi} = 0$. Together with the formula for the degree (2.10) this implies

$$E_{\text{gauge}}[\boldsymbol{\phi}, A_1, A_2] \geq 4\pi|\deg[\boldsymbol{\phi}]|, \tag{2.12}$$

with equality if and only if one of the Bogomol'nyi equations holds:

$$\begin{aligned} D_1\boldsymbol{\phi} &= \mp\boldsymbol{\phi} \times D_2\boldsymbol{\phi} \\ F_{12} &= \pm(1 - \mathbf{n}\cdot\boldsymbol{\phi}). \end{aligned} \tag{2.13}$$

It is instructive to write these equations also in a different form which results when the target space S^2 is stereographically projected onto $\mathbf{C} \cup\{\infty\}$. More precisely defining a complex-valued field $u = u_1 + iu_2$ via

$$u_1 = \frac{\phi_1}{1 + \phi_3} \qquad u_2 = \frac{\phi_2}{1 + \phi_3}, \tag{2.14}$$

the Eqs. (2.13) become, in terms of u,

$$\begin{aligned} D_1 u &= \mp iD_2 u \\ F_{12} &= \pm\frac{2|u|^2}{1 + |u|^2}, \end{aligned} \tag{2.15}$$

where D_j now stands for $\partial_j + iA_j$, $j = 1, 2$. These equations imply the following second order equation for $\varphi = \ln u$:

$$\Delta(\varphi + \bar{\varphi}) = \frac{4}{1 + e^{-(\varphi+\bar{\varphi})}}. \tag{2.16}$$

The integrability of such "non-linear Laplace equations" has been studied in the literature, but the present equation lies outside a small class of such equations which are known to be integrable by standard methods, such as scattering transforms [8]. To find solutions of the Bogomol'nyi equations (2.13) we therefore resort to numerical methods.

3 Solving the Bogomol'nyi equations

When seeking solutions of Eq. (2.13) with non-zero degree we restrict attention to fields which are invariant under simultaneous rotations and reflections in space and target space. Thus we assume that $\boldsymbol{\phi}$ is of the so-called hedgehog form

$$\boldsymbol{\phi}(\boldsymbol{x}) = (\sin f(r) \cos N\theta, \sin f(r) \sin N\theta, \cos f(r)), \tag{3.1}$$

where (r, θ) are polar coordinates in the $\boldsymbol{x}$-plane, N is a non-zero integer and f is a function satisfying certain boundary conditions to be specified below. The gauge field is assumed to have only a θ-component which is of the form

$$A_\theta = Na(r). \tag{3.2}$$

(For a more detailed justification of this ansatz see [9].) Then the magnetic component of the field strength is simply

$$F_{12} = N\frac{a'}{r}. \tag{3.3}$$

To obtain fields which are regular at the origin we require

$$f(0) = \pi \quad \text{and} \quad a(0) = 0, \tag{3.4}$$

and to ensure also that the energy is finite we impose

$$\lim_{r\to\infty} f(r) = 0 \quad \text{and} \quad \lim_{r\to\infty} a'(r) = 0\,. \tag{3.5}$$

One checks that the degree of such a configuration is $-N$.

The Bogomol'nyi equations (2.13) imply the following coupled first order differential equation for f and a:

$$f' = -|N|\frac{a+1}{r} \sin f \tag{3.6}$$

$$a' = -\frac{r}{|N|}(1 - \cos f)\,, \tag{3.7}$$

where the alternative signs of (2.13) have been absorbed into the modulus sign. For brevity we will refer to the boundary value problem posed by these differential equations together with the boundary conditions (3.4) and (3.5) as BVP. As a first step in its discussion we establish the

Proposition: *The boundary value problem BVP has no solution if* $|N| = 1$, *but it has a one-parameter family of solutions if* $|N| > 1$.

Before entering the proof we note the solutions of Eqs. (3.6) and (3.7) for small r. Using the boundary condition (3.4) and keeping only the leading powers in r one finds

$$f \approx \pi + Ar^{|N|} \quad \text{and} \quad a \approx -\frac{1}{|N|}r^2, \tag{3.8}$$

where A is an *arbitrary* constant. When integrating Eqs. (3.6) and (3.7) numerically one cannot start the integration at the regular-singular point $r = 0$. Instead we integrate from some small value ($r = 10^{-6}$ in practice) outwards, imposing the initial values there according to (3.8). According to the proposition, there is a family of values for A that will lead to a solution satisfying the boundary conditions (3.5) at infinity when $|N| > 1$ but there is no such value when $|N| = 1$.

The proof of the proposition proceeds in four steps; in it a function g is called increasing (decreasing) if $x > y \Rightarrow g(x) \geq g(y)$ ($x > y \Rightarrow g(x) \leq g(y)$).

1. It is clear from Eq. (3.7) that $a' \leq 0$, i.e. a is a decreasing function. Moreover, since $0 \leq (1 - \cos f) \leq 2$, the boundary conditions (3.4) and the intermediate value theorem imply

$$-\frac{r^2}{|N|} \leq a < 0. \tag{3.9}$$

2. For any solution of BVP, $f(r) \in (0, \pi]$ for all r. To prove this we first show that $f(r) > 0$ for all r. For suppose that f were close to zero, so that Eqs. (3.6) and (3.7) simplify to

$$f' = -|N|\frac{a+1}{r}f \tag{3.10}$$

$$a' = -\frac{r}{2|N|}f^2. \tag{3.11}$$

Then, from the first equation $\int d \ln f = -|N| \int dr\, (a + 1)/r$. However, while the left hand side diverges as $f \to 0$, the right hand side is finite for any finite interval of integration (we can assume without loss of generality that $r = 0$ is not included in the interval). Thus f cannot vanish for any finite value of r; since it is continuous and non-zero for $r = 0$ it is positive for all r. By a similar argument one shows that if $f(r_0) > \pi$ for some $r_0 > 0$ then $f(r) > \pi$ for all $r > r_0$, which violates the boundary condition (3.5). Thus $0 < f \leq \pi$ as claimed.

3. For any solution of BVP, $a(r) > -1$ for all r. Suppose this were not the case. Then there is an r_1 such that $a(r_1) \leq -1$ and, since a is decreasing, $a(r) \leq -1$ for all $r > r_1$.

However, since $f \in (0, \pi]$, Eq. (3.6) then implies that f is increasing for $r > r_1$, which is incompatible with the boundary condition (3.5). Combining this result with the inequality (3.9) we conclude $-1 < a < 0$. Note that it then follows from Eq. (3.6) that f is a decreasing function.

4. Since a is bounded below by -1 and decreasing the limit $\lim_{r\to\infty} a(r) = \alpha$ exists and $\alpha \in [-1, 0)$. For large r, and hence small f, Eq. (3.10) holds and implies that f is asymptotic to $Cr^{-|N|(\alpha+1)}$, where C is some positive constant. However, Eq. (3.11) then tells us that for large r, $a' \approx -(C^2/2|N|)r^{(1-2|N|(\alpha+1))}$. Thus a can only converge if $2|N|(\alpha + 1) - 1 > 1$, or equivalently

$$\alpha > \frac{1}{|N|} - 1. \tag{3.12}$$

This is impossible to satisfy if $|N| = 1$, but for $|N| > 1$ there is whole interval $(-1+1/|N|, 0)$ of acceptable asymptotic values for a, and hence there is a one-parameter family of solutions of BVP, which was to be shown.

When $|N| > 1$ the variable parametrising the solutions of BVP can be taken to be α or, more physically, the total magnetic flux Φ which is related to α via

$$\Phi = \int d^2x\, F_{12} = N \int d\theta\, \alpha = 2\pi N\alpha. \tag{3.13}$$

This formula shows in particular that the magnetic flux is not quantised. Mathematically this means that the gauge field (A_1, A_2) of solutions of BVP does not extend to the compactification $\mathbf{R}^2 \cup \{\infty\}$; for if it did, $(\Phi/2\pi)$ would be the first Chern number of a $U(1)$ bundle over a compact manifold, which is necessarily an integer. The magnetic flux is nonetheless an interesting quantity to consider because it is conserved if one rules out infinite energy configurations. This follows from Faraday's law of induction

$$\frac{d\Phi}{dt} = \oint_C \boldsymbol{E}\cdot d\boldsymbol{l}, \tag{3.14}$$

where the contour C is the circle at infinity. The integral on the right hand side is only non-zero if the electric field $\boldsymbol{E}$ falls off for large r no faster than $1/r$, which is precisely the condition for the electric field to have infinite energy.

We have numerically solved BVP with $N = 2$ for various values of Φ in the allowed range $(-2\pi, 0)$. Since the qualitative features of the functions f and a are clear from the proof of the proposition we only show plots of the energy density and the magnetic field of the solutions. The energy density is maximal on a ring whose radius is another useful measure of the soliton size. As the magnitude of the magnetic flux increases this radius increases and reaches a finite limit for $|\Phi| \to 2\pi$. In Figs. 1 and 2 the energy density and the magnetic field for a soliton whose (modulus of the) magnetic flux is close to that limit are drawn with a solid line. In the limit $|\Phi| \to 0$ the soliton's size becomes arbitrarily small.

4 Discussion and outlook

We have studied rotationally symmetric solutions of the Bogomol'nyi equations in a gauged version of the $O(3)$ sigma model of degree N with $|N| \neq 1$. There is no finite energy solution of degree 1, but there are probably many more solutions of degree $N > 1$ than considered here. As in other field theories of Bogomol'nyi type this can presumably be shown using an index theorem and a vanishing theorem for an appropriate Dirac operator. Typically, there is a whole manifold of degree N solutions of the Bogomol'nyi equation (called a moduli space), and the dimension of this manifold is a linear function of N.

Solitons of Bogomol'nyi type which display all these properties and which are rather similar to the solitons discussed here are the Q-lumps mentioned in the introduction. Q-lumps of degree 1 necessarily have infinite energy, but there exists a $(4N-2)$-dimensional family of Q-lumps of degree $N > 1$. These include configurations which are made up of N well-separated single Q-lumps. Similarly there should be solutions of degree $N > 1$ in the present model whose energy density is peaked at N points in the plane. These could then be interpreted as superpositions of N solitons of degree 1; in this sense, solitons of degree 1 can exist as part of a multisoliton configuration.

Like Q-lumps the solitons discussed here can have an arbitrary size, with the role of the size parameter being played by the magnetic flux. However, whereas the energy of a Q-lump varies with the Q-lump's size, the energy of the solitons discussed here is degenerate with respect to changes in the magnetic flux. Thus the scaling degeneracy of the pure $O(3)$ sigma model persists in the present model in a mutated form. There is an important difference, however. In the $O(3)$ sigma model, solitons exhibit a "rolling instability", in the sense that under a small perturbation they either shrink to a thin spike or expand without limit (there is a subtlety here: in the so-called moduli space approximation such scale changes require infinite energy for single solitons and most multisolitons; numerical simulations, however, suggest that they do occur in the full field theory). In the present model, by contrast, there is a completely general argument - Faraday's law of induction - according to which the total flux of a configuration can only change at the cost of infinite energy.

The constancy of the total flux does not prevent individual solitons in a multisoliton configuration from shrinking into thin spikes. It would be interesting to see if this happens as a consequence of soliton interactions. If the above conjecture about moduli spaces for solitons of degree $N > 1$ is correct this question could be investigated using the moduli space approximation to soliton dynamics.

Finally we return to the question of stabilising solitons in the $O(4)$ sigma model in (3+1) dimensions with a Maxwell term. This would lead to the "Skyrme-Maxwell model without a Skyrme term" mentioned in the introduction. Of course it is well-known that topologically non-trivial gauge fields can stabilise solitons in 2 or 3 spatial dimensions, like in the case of abelian Higgs vortices or t'Hooft-Polyakov magnetic monopoles. However, the analysis of the gauged $O(3)$ sigma model here indicates that such a stabilisation is also possible if there is only a dynamical reason why the gauge field cannot vanish during time evolution. Thus one could try to stabilise a Skyrmion in three spatial dimensions via electric charge. The electric charge is conserved and, if it is non-zero, produces an electric field which prevents the Skyrmion from collapsing to zero size. It seems quite possible that there are such electrically charged Skyrmion solutions in (3+1)-dimensional "Skyrme-Maxwell theory without at Skyrme term". The basic idea is to balance the tendency of the scalar field to collapse by the electrostatic repulsion of like charges.

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Note added

Trevor Samols has recently shown that for any solution of the Bogomol'nyi equations (2.15) of degree $N > 1$, the magnetic flux lies in the range $0 < (\Phi/2\pi) \leq (N-1)$. Moreover he proved that for fixed degree $N > 1$ and magnetic flux in the above range, there is a $(4N - 2 - 2\{\Phi/2\pi\})$-parameter family of solutions of (2.15), were $\{\Phi/2\pi\}$ is the smallest integer larger than $(\Phi/2\pi)$.

References

[1] A.A. Belavin and A.M. Polyakov, JETP Lett. 22 (1975) 245.

[2] R.A. Leese, M. Peyrard and W.J. Zakrzewski, Nonlinearity 3 (1990) 387.

[3] M. Peyrard, B.M.A.G. Piette and W.J. Zakrzewski, Nonlinearity 5 (1992) 563-583 and 585-600;
B.M.A.G. Piette, B.J. Schroers and W.J. Zakrzewski, Nucl.Phys. B439 (1995) 205.

[4] R.A. Leese, Nucl.Phys. B344 (1990) 33.

[5] R.A. Leese, Nucl.Phys. B366 (1991) 283.

[6] A. Jaffe and C. Taubes, Vortices and Monopoles (Birkhäuser, Boston, 1980) p.1.

[7] R. Jackiw, K. Lee and E.J. Weinberg, Phys.Rev.D 42 (1990) 3488.

[8] A.P Fordy and J. Gibbons, Commun.Math.Phys. 77 (1980) 21.

[9] J. Gladikowski, B.M.A.G. Piette and B.J. Schroers, Durham preprint DTP 95-29 (1995).

Figure Captions

Fig. 1. The energy density as a function of r for $N = 2$ solitons with $(\Phi/2\pi) = -0.96$ (solid line), $(\Phi/2\pi) = -0.43$ (dashed line) $(\Phi/2\pi) = -0.18$ (dashed-dotted line). The function e plotted here is the integrand of (2.7) divided by 4π.

Fig. 2. The magnitude of the magnetic field F_{12} as a function of r for $N = 2$ solitons with $(\Phi/2\pi) = -0.96$ (solid line), $(\Phi/2\pi) = -0.43$ (dashed line) $(\Phi/2\pi) = -0.18$ (dashed-dotted line).